Some Surprising Conservative and Nonconservative Moments in the Dynamics of Rods and Rigid Bodies

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Abstract Representations for conservative and nonconservative moments in classical mechanics are discussed in this expository article. When the rotation is parameterized by a set of Euler angles, a particularly transparent representation can be found which has ties to classic works in mechanics dating to Lagrange in 1780 and joint coordinate systems that are commonly used in orthopaedic biomechanics. The article also surveys connections between Lagrange's equations of motion and the Newton-Euler equations of motion. A variant on the Lagrange top and a satellite dynamics problem are presented to illustrate some of the key concepts discussed in the paper.

1 Introduction

In a remarkable paper, Lagrange (1780) presented a dynamic model to explain the oscillations (librations) in the attitude of the Moon as seen by an Earth-based observer. The starting point for his model featured for the first time his celebrated equations of motion (cf. Lagrange (1780, Section 11)):

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^K}\right) - \frac{\partial T}{\partial q^K} = -\frac{\partial V}{\partial q^K}, \qquad (K = 1, \ldots, 6)\,, \tag{1}$$

where V and T are the respective potential and kinetic energies of the Moon. Later in this work (cf. Lagrange (1780, Section 21)), Lagrange used a set of 3-1-3 Euler angles to parametrize the rotation of the Moon.

A few years prior to Lagrange's work, Euler published a series of seminal works on the dynamics of rigid bodies (cf. Euler (1752, 1775)). Among the contributions from Euler's works that have had a lasting influence are the

Euler angle parameterization of rotations and the Newton-Euler equations of motion for a rigid body of mass m:

$$\mathbf{F} = m\ddot{\bar{\mathbf{x}}}, \qquad \mathbf{M} = \dot{\mathbf{H}}, \qquad \mathbf{H} = \mathbf{J}\boldsymbol{\omega}, \tag{2}$$

where $\mathbf{F}$ is the resultant force acting on the rigid body, $\mathbf{M}$ is the resultant moment relative to the center of mass of the rigid body, $\mathbf{J}$ is the moment of inertia tensor of the rigid body relative to its center of mass, $\bar{\mathbf{x}}$ is the position vector of the center of mass, $\mathbf{H}$ is the angular momentum relative to the center of mass of the rigid body, and $\boldsymbol{\omega}$ is the angular velocity vector of the rigid body.

Lagrange's treatment of the libration problem is surprising. First, although he employs Euler angles and is completely comfortable with mass moments of inertia, he uses an entirely different formulation of the problem than the Newton-Euler form (2). While he calculates gravitational forces and approximations to the gravitational potential energy, it is not clear what the corresponding gravitational moments are from his work. Indeed the moment in question, known as a gravity-gradient torque (cf. Eqn. $(83)_2$), is credited to James Mac Cullagh (1809–1847) following the posthumous publication of his lecture notes by Allman (1855).

Assuming that the dynamic equations of motion formulated using Lagrange's equations (1) and the Newton-Euler equations (2) are equivalent, it is natural to ask if some of the partial derivatives $-\frac{\partial V}{\partial q^K}$ can be interpreted as moment components. Using results on dual Euler basis vectors from O'Reilly (2007) we are able to show how these partial derivatives are specific components of force and moment vectors and thus facilitate physical interpretations of the partial derivatives $-\frac{\partial V}{\partial q^K}$. Our discussion is illuminated with examples from rigid body dynamics and orthopaedic biomechanics and also highlights the simplest known representation for a conservative moment.

An outline of this expository article is as follows. Background on the Euler angle parameterization of a rotation is collected in Section 2. The notation we use for the three angles follows Lagrange (1780). We supplement his treatment with a discussion of the Euler basis and dual Euler basis vectors and the representation of vectors using these distinct bases. In Section 3, the relationship between Lagrange's equations of motion and the Newton-Euler equations of motion for a single rigid body are examined. Particular attention is paid to the incorporation of ideal integrable constraints and potential energies. A simple representation for a conservative moment that features the dual Euler basis vectors is discussed in Section 4. To dispel possible confusion, the case where the motion of a rigid body is constrained to have a fixed axis of rotation is presented in Section 5. We close the paper with two examples. First, a derivation of the equations of motion of a Lagrange top subject to applied forces and moments is discussed. Then, in Section 7, we return to the problem of a satellite in a gravitational field that was the subject of Lagrange (1780). Among other matters, we are able to demonstrate how the partial derivatives $-\frac{\partial V}{\partial q^K}$ in Eqn. (1) can be considered as components of a moment vector.

The notation and terminology employed in this paper closely follows the textbook by O'Reilly (2008). We appeal to a recent expository article by O'Reilly and Srinivasa (2014) when discussing constraint forces and constraint moments. Additional complementary background on rotations can be found in the exceptional survey article by Shuster (1993) and the online resource http://rotations.berkeley.edu/.

2 Background on Euler Angles and Bases

Central to our discussion is the method of Euler angles to parameterize a rotation tensor. Of the twelve available sets of Euler angles, we focus our attention here on the 3-1-3 set. We define a fixed right-handed orthonormal basis $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ and use the rotation tensor $\mathbf{Q}$ to define a basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$:

$$\mathbf{e}_i = \mathbf{Q}\mathbf{E}_i. \tag{3}$$

It is straightforward to show using the facts that $\mathbf{Q}\mathbf{Q}^T = \mathbf{I}$ and $\det(\mathbf{Q}) = 1$ that the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is right-handed and orthonormal. In addition, the rotation tensor $\mathbf{Q}$ has the representations

$$\mathbf{Q} = \sum_{k=1}^{3} \mathbf{e}_k \otimes \mathbf{E}_k = \sum_{i=1}^{3}\sum_{k=1}^{3} Q_{ik}\mathbf{E}_i \otimes \mathbf{E}_k = \sum_{i=1}^{3}\sum_{k=1}^{3} Q_{ik}\mathbf{e}_i \otimes \mathbf{e}_k. \tag{4}$$

That is, the components $Q_{ik} = \mathbf{e}_k \cdot \mathbf{E}_i$ of $\mathbf{Q}$ can be considered as direction cosines.

In the 3-1-3 set of Euler angles, the tensor $\mathbf{Q}$ is decomposed into the product of three simple rotations:

$$\mathbf{Q} = \mathbf{Q}_{\mathbf{E}}(\varphi, \mathbf{g}_3)\, \mathbf{Q}_{\mathbf{E}}(\omega, \mathbf{g}_2)\, \mathbf{Q}_{\mathbf{E}}(\psi, \mathbf{g}_1), \tag{5}$$

where the function $\mathbf{Q}_{\mathbf{E}}(\theta, \mathbf{i})$ describes a rotation about an axis described by a unit vector $\mathbf{i}$ through a counterclockwise angle θ:

$$\mathbf{Q}_{\mathbf{E}}(\theta, \mathbf{i}) = \cos(\theta)\,(\mathbf{I} - \mathbf{i} \otimes \mathbf{i}) + \sin(\theta)\text{skew}(\mathbf{i}) + \mathbf{i} \otimes \mathbf{i}. \tag{6}$$

In the above representation for a rotation tensor, the operator $\text{skew}(\mathbf{i})$ transforms $\mathbf{i}$ into a skew-symmetric tensor such that $\mathbf{i} \times \mathbf{b} = \text{skew}(\mathbf{i})\mathbf{b}$ for any vector $\mathbf{b}$. The associated operator ax transforms a skew-symmetric tensor into a vector: $\text{ax}(\mathbf{A}) \times \mathbf{b} = \mathbf{Ab}$ where $\mathbf{A} = -\mathbf{A}^T$ is a skew-symmetric tensor.

The basis $\{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\}$ is known as the Euler basis. This basis is not orthogonal, and, for the 3-1-3 set of interest here,

$$\begin{aligned}
\mathbf{g}_1 &= \mathbf{E}_3 \\
&= \cos(\omega)\,\mathbf{e}_3 + \sin(\omega)\left(\cos(\varphi)\,\mathbf{e}_2 + \sin(\varphi)\,\mathbf{e}_1\right), \\
\mathbf{g}_2 &= \cos(\psi)\,\mathbf{E}_1 + \sin(\psi)\,\mathbf{E}_2 \\
&= \cos(\varphi)\,\mathbf{e}_1 - \sin(\varphi)\,\mathbf{e}_2, \\
\mathbf{g}_3 &= \cos(\omega)\,\mathbf{E}_3 + \sin(\omega)\left(\sin(\psi)\,\mathbf{E}_1 - \cos(\psi)\,\mathbf{E}_2\right) \\
&= \mathbf{e}_3.
\end{aligned} \tag{7}$$

The angles ψ and φ range from 0 to 2π. Because

$$(\mathbf{g}_1 \times \mathbf{g}_2) \cdot \mathbf{g}_3 = -\sin(\omega), \tag{8}$$

in order to ensure that the Euler basis is a basis for $\mathbb{E}^3$, we restrict the second angle $\omega \in (0, \pi)$. Other perspectives on the singularity when $\omega = 0, \pi$ include noting that ψ and ω are polar coordinates for the axis of rotation $\mathbf{g}_3 = \mathbf{e}_3$. Thus, the singularity arises when multiple values of $\psi + \varphi$ (when $\omega = \pi$) and $\psi - \varphi$ (when $\omega = 0$) are possible for a given rotation tensor $\mathbf{Q}$.

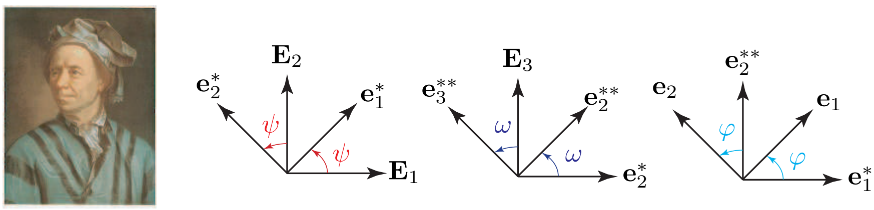

Figure 1: Schematic of a set of 3-1-3 Euler angles that are used to parameterize a rotation from the basis $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ to the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. The three axes of rotation are $\mathbf{g}_1 = \mathbf{E}_3 = \mathbf{e}_3^*$, $\mathbf{g}_2 = \mathbf{e}_1^* = \mathbf{e}_1^{**}$, and $\mathbf{g}_3 = \mathbf{e}_3^{**} = \mathbf{e}_3$ and the inset image is Handmann's portrait of Leonhard Euler (1707–1783) from 1753. For details on the intermediate bases used to construct the figure, see Eqn. (9).

Referring to Figure 1, it is straightforward to transform from $\mathbf{e}_i$ to $\mathbf{E}_i$ and vice versa with the help of two pairs of intermediate bases $\{\mathbf{e}_1^*, \mathbf{e}_2^*, \mathbf{e}_3^*\}$

and $\{\mathbf{e}_1^{**}, \mathbf{e}_2^{**}, \mathbf{e}_3^{**}\}$:

$$\begin{aligned}
\begin{bmatrix} \mathbf{e}_1^{*} \\ \mathbf{e}_2^{*} \\ \mathbf{e}_3^{*} \end{bmatrix} &= \begin{bmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}, \\
\begin{bmatrix} \mathbf{e}_1^{**} \\ \mathbf{e}_2^{**} \\ \mathbf{e}_3^{**} \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\omega) & \sin(\omega) \\ 0 & -\sin(\omega) & \cos(\omega) \end{bmatrix} \begin{bmatrix} \mathbf{e}_1^{*} \\ \mathbf{e}_2^{*} \\ \mathbf{e}_3^{*} \end{bmatrix}, \\
\begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix} &= \begin{bmatrix} \cos(\varphi) & \sin(\varphi) & 0 \\ -\sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{e}_1^{**} \\ \mathbf{e}_2^{**} \\ \mathbf{e}_3^{**} \end{bmatrix}.
\end{aligned} \tag{9}$$

We note that the three matrices in Eqn. (9) can be combined to provide expressions for the components $Q_{ik} = (\mathbf{Q}\mathbf{E}_k) \cdot \mathbf{E}_i$ of $\mathbf{Q}$:

$$\begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} = \begin{bmatrix} -s_1c_2s_3 + c_1c_3 & -c_1s_3 - c_3c_2s_1 & s_1s_2 \\ c_2c_1s_3 + c_3s_1 & -s_1s_3 + c_2c_1c_3 & -c_1s_2 \\ s_3s_2 & s_2c_3 & c_2 \end{bmatrix}. \tag{10}$$

In writing expressions for the components Q_{ik}, we have used the helpful abbreviations $c_1 = \cos(\psi)$, $s_1 = \sin(\psi)$, $c_2 = \cos(\omega)$, $s_2 = \sin(\omega)$, $c_3 = \cos(\varphi)$, and $s_3 = \sin(\varphi)$.

The Euler basis vectors feature in the representation of the angular velocity vector $\boldsymbol{\omega}$ associated with the rotation tensor $\mathbf{Q}$. In particular,

$$\begin{aligned}
\boldsymbol{\omega} = \text{ax}\left(\dot{\mathbf{Q}}\mathbf{Q}^T\right) &= \dot{\varphi}\mathbf{g}_3 + \dot{\omega}\mathbf{g}_2 + \dot{\psi}\mathbf{g}_1 \\
&= \left(\dot{\psi}\sin(\omega)\sin(\varphi) + \dot{\omega}\cos(\varphi)\right)\mathbf{e}_1 \\
&+ \left(\dot{\psi}\sin(\omega)\cos(\varphi) - \dot{\omega}\sin(\varphi)\right)\mathbf{e}_2 + \left(\dot{\psi}\cos(\omega) + \dot{\varphi}\right)\mathbf{e}_3.
\end{aligned} \tag{11}$$

This representation can be established by direct, but lengthy computation of $\dot{\mathbf{Q}}$ using Eqn. (10) or, more rapidly, using two relative angular velocity vectors as in Casey and Lam (1986).

2.1 The Euler and Dual Euler Bases

In addition to the Euler basis, we also have a companion dual Euler basis. Given an Euler basis, the corresponding dual Euler basis vectors are defined by the nine relations

$$\mathbf{g}^k \cdot \mathbf{g}_i = \delta_i^k, \qquad (i = 1, 2, 3, \text{ and } k = 1, 2, 3), \tag{12}$$

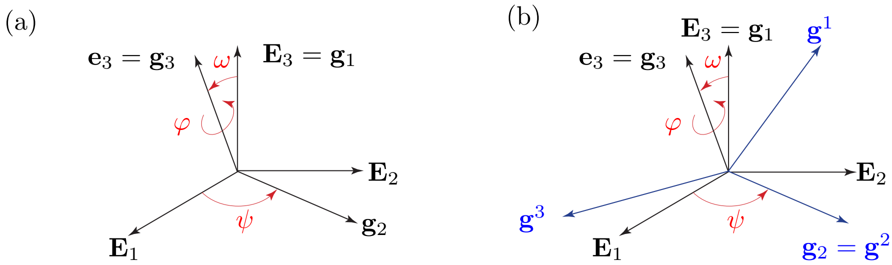

Figure 2: Schematic of the Euler and dual Euler basis vectors associated with a 3-1-3 set of Euler angles. (a) The Euler basis vectors, $\mathbf{g}_1$, $\mathbf{g}_2$, and $\mathbf{g}_3$, and their relation to the Euler angles (cf. Eqn. (7)) and (b) the corresponding dual Euler basis vectors, $\mathbf{g}^1$, $\mathbf{g}^2$, and $\mathbf{g}^3$ (cf. Eqn. (14)).

where δ_i^k is the Kronecker delta: $\delta_i^k = 1$ if $i = k$ and is otherwise 0. The solution to these nine equations is known in differential geometry:[1]

$$\begin{aligned}
\mathbf{g}^1 &= \frac{\mathbf{g}_2 \times \mathbf{g}_3}{(\mathbf{g}_1 \times \mathbf{g}_2) \cdot \mathbf{g}_3}, \\
\mathbf{g}^2 &= \mathbf{g}_2, \\
\mathbf{g}^3 &= \frac{\mathbf{g}_1 \times \mathbf{g}_2}{(\mathbf{g}_1 \times \mathbf{g}_2) \cdot \mathbf{g}_3}.
\end{aligned} \tag{13}$$

The expression for $\mathbf{g}^2$ is greatly simplified because the Euler basis vector $\mathbf{g}_2$ is perpendicular to the other two: $\mathbf{g}_1 \perp \mathbf{g}_2$ and $\mathbf{g}_3 \perp \mathbf{g}_2$. With the help of Eqn. (13), we now compute the dual Euler basis vectors for the 3-1-3 set of Euler angles:

$$\begin{aligned}
\mathbf{g}^1 &= \operatorname{cosec}(\omega)\left(\sin(\varphi)\,\mathbf{e}_1 + \cos(\varphi)\,\mathbf{e}_2\right), \\
\mathbf{g}^2 &= \cos(\varphi)\,\mathbf{e}_1 - \sin(\varphi)\,\mathbf{e}_2, \\
\mathbf{g}^3 &= \cot(\omega)\left(-\cos(\varphi)\,\mathbf{e}_2 - \sin(\varphi)\,\mathbf{e}_1\right) + \mathbf{e}_3.
\end{aligned} \tag{14}$$

The Euler and dual Euler bases are sketched in Figure 2.

We observe from Eqn. (14) that the dual Euler basis vectors are not defined when $\omega = 0, \pi$. The dual Euler basis vectors were first introduced

[1] We are exploiting the correspondence between the Euler and dual Euler basis sets and covariant and contravariant sets of basis vectors in differential geometry and are able to use a well-known result. See, e.g., Green and Zerna (1968, Eqn. (1.9.13)) or Simmonds (1982, Exercise 2.11)).

in O'Reilly and Srinivasa (2002) and O'Reilly (2007). They are also related to the dual vectors described by Howard et al. (1998) and Žefran and Kumar (2002) in their discussion of screw motions for rigid bodies.

2.2 Vector Representations

As discussed in Nichols and O'Reilly (2017) and O'Reilly et al. (2013), the dual Euler basis feature in representations for the joint moment vector that is commonly used in orthopaedic biomechanics. To elaborate, a vector $\mathbf{b}$ has multiple representations including

$$\begin{aligned}\mathbf{b} &= b^1\mathbf{g}_1 + b^2\mathbf{g}_2 + b^3\mathbf{g}_3 \\ &= b_1\mathbf{g}^1 + b_2\mathbf{g}^2 + b_3\mathbf{g}^3,\end{aligned} \tag{15}$$

where

$$b^k = \mathbf{b}\cdot\mathbf{g}^k, \qquad b_k = \mathbf{b}\cdot\mathbf{g}_k. \tag{16}$$

Referring to Figure 3, in biomechanics of anatomical joints, the first and third Euler basis vectors are identified with landmarks on the respective bones and moment components $\mathbf{M}\cdot\mathbf{g}_k$ are computed.[2] Consequently, in order to reconstruct the moment vector, the dual Euler basis vectors are needed: $\mathbf{M} = \sum_{k=1}^{3}\left(\mathbf{M}\cdot\mathbf{g}_k\right)\mathbf{g}^k$.

3 Lagrange's Equations of Motion and the Newton-Euler Equations of Motion

Consider a rigid body of mass m which has an inertia tensor $\mathbf{J}$ relative to its center of mass. We assume that a set of six coordinates are used to characterize the kinematics of the rigid body:

$$\begin{aligned}\bar{\mathbf{x}} &= \bar{\mathbf{x}}\left(q^1,\ldots,q^6\right), \\ \mathbf{Q} &= \mathbf{Q}\left(q^1,\ldots,q^6\right).\end{aligned} \tag{17}$$

These coordinates are usually chosen to readily accommodate the integrable constraints on the system. Referring to Figure 4, for the Lagrange top q^1, q^2, and q^3 are usually chosen to be a set of Euler angles while q^4, q^5, and q^6 are chosen to be the Cartesian coordinates of the fixed point O of the top. In satellite dynamics problems where a steady motion of the satellite involves a 1-1 locking of the orbital angular speed and the angular velocity,

[2]See, for example, Desroches et al. (2010), Grood and Suntay (1983), and Schache and Baker (2007).

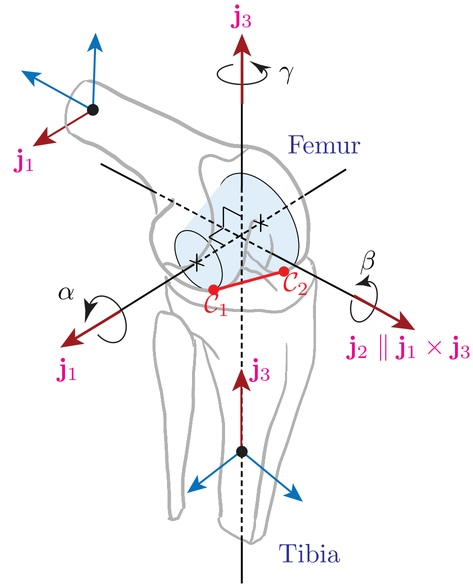

Figure 3: Schematic of a joint coordinate system for the human knee joint. The axis $\mathbf{j}_1$ corotates with the femur and $\mathbf{j}_3$ corotates with the tibia. The angles α, β, γ and the axes $\{\mathbf{j}_1, \mathbf{j}_2, \mathbf{j}_3\}$ can be identified with a set of 3-2-1 (or 1-2-3) Euler angles and their associated Euler basis vectors.

a set of cylindrical polar coordinates are chosen for $q^1, q^2 = \vartheta$, and q^3 and the rotation is parameterized by a set of Euler angles and the polar angle: $\mathbf{Q} = \mathbf{Q}\left(\vartheta, q^4 = \psi, q^5 = \omega, q^6 = \varphi\right)$.

It is straightforward to show that

$$\bar{\mathbf{v}} = \sum_{K=1}^{6} \dot{q}^K \frac{\partial \bar{\mathbf{x}}}{\partial q^K}, \qquad \frac{\partial \bar{\mathbf{x}}}{\partial q^K} = \frac{\partial \bar{\mathbf{v}}}{\partial \dot{q}^K}. \tag{18}$$

Further,

$$\boldsymbol{\omega} = \sum_{K=1}^{6} \dot{q}^K \mathbf{w}_K, \qquad \mathbf{w}_K = \frac{\partial \boldsymbol{\omega}}{\partial \dot{q}^K} = \operatorname{ax}\left(\frac{\partial \mathbf{Q}}{\partial q^K}\mathbf{Q}^T\right). \tag{19}$$

If a set of Euler angles are used to parameterize $\mathbf{Q}$,

$$\mathbf{Q} = \mathbf{Q}\left(q^6 = \varphi, q^5 = \omega, q^4 = \psi\right), \tag{20}$$

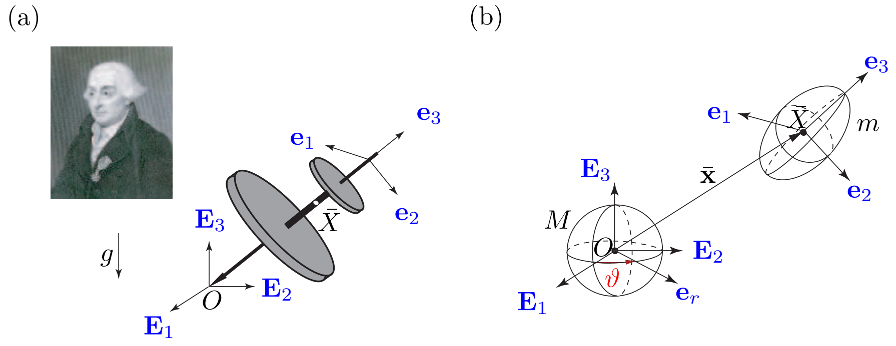

Figure 4: (a) Schematic of a rigid body freely rotating about a fixed point O. This rigid body is commonly known as a Lagrange top and the inset image is a portrait of Joseph-Louis Lagrange (1736–1813). (b) Schematic of a rigid body of mass m in motion about a fixed rigid body of mass M.

then $\mathbf{w}_{3+k} = \mathbf{g}_k$ and $\mathbf{w}_{1,2,3} = \mathbf{0}$.

The kinetic energy of the rigid body has the representation

$$T = \frac{m}{2}\bar{\mathbf{v}} \cdot \bar{\mathbf{v}} + \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{J}\boldsymbol{\omega}. \tag{21}$$

Typically the corotational (or body-fixed) basis vectors $\mathbf{e}_i$ are chosen to be the principal axis of the body. In this case, $\mathbf{J}$ can be expressed as

$$\mathbf{J} = \lambda_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 \otimes \mathbf{e}_2 + \lambda_3 \mathbf{e}_3 \otimes \mathbf{e}_3, \tag{22}$$

where λ_k are the principal mass moments of inertia.

It can be shown that Lagrange's equations of motion are equivalent to a linear combination of the Newton-Euler balance laws:[3]

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^K}\right) - \frac{\partial T}{\partial q^K} = \mathbf{F} \cdot \frac{\partial \bar{\mathbf{v}}}{\partial \dot{q}^K} + \mathbf{M} \cdot \frac{\partial \boldsymbol{\omega}}{\partial \dot{q}^K}. \tag{23}$$

That is,

$$\begin{aligned} \frac{\partial T}{\partial \dot{q}^K} &= m\bar{\mathbf{v}} \cdot \frac{\partial \bar{\mathbf{v}}}{\partial \dot{q}^K} + \mathbf{H} \cdot \frac{\partial \boldsymbol{\omega}}{\partial \dot{q}^K}, \\ \frac{\partial T}{\partial q^K} &= m\bar{\mathbf{v}} \cdot \frac{d}{dt}\left(\frac{\partial \bar{\mathbf{v}}}{\partial \dot{q}^K}\right) + \mathbf{H} \cdot \frac{d}{dt}\left(\frac{\partial \boldsymbol{\omega}}{\partial \dot{q}^K}\right). \end{aligned} \tag{24}$$

[3] A proof of this correspondence can be found in Casey (1995). Casey's proof is discussed in the textbook O'Reilly (2008) where additional examples are presented.

It is an interesting exercise to establish these results first for a single particle of mass m - where the angular momentum terms can be ignored. Indeed this case is discussed in the classic textbook Synge and Griffith (1959).

3.1 A Force $\mathbf{F}_A$ Acting at a Point A

The right hand side of Lagrange's equations (23) have several simplifications. First, suppose that a force $\mathbf{F}_A$ acts at a point X_A which has a position vector $\mathbf{x}_A$ and velocity vector $\mathbf{v}_A$:

$$\mathbf{v}_A = \bar{\mathbf{v}} + \boldsymbol{\omega} \times (\mathbf{x}_A - \bar{\mathbf{x}}) \, . \tag{25}$$

The position vectors $\mathbf{x}_A$ and $\bar{\mathbf{x}}$ can be expressed as functions of the six coordinates $q^1, \ldots, q^6$. Unlike $\bar{\mathbf{v}}$, $\boldsymbol{\omega}$, and $\mathbf{v}_A$, this pair of position vectors do not depend on the velocities $\dot{q}^1, \ldots, \dot{q}^6$.

For the force $\mathbf{F}_A$, a simple differentiation of Eqn. (25) with respect to $\dot{q}^K$ can be used to show that

$$\mathbf{F}_A \cdot \frac{\partial \bar{\mathbf{v}}}{\partial \dot{q}^K} + ((\mathbf{x}_A - \bar{\mathbf{x}}) \times \mathbf{F}_A) \cdot \frac{\partial \boldsymbol{\omega}}{\partial \dot{q}^K} = \mathbf{F}_A \cdot \frac{\partial \mathbf{v}_A}{\partial \dot{q}^K}. \tag{26}$$

This identity is helpful in several respects. First, it enables a direct comparison of treatments of Lagrange's equations where a virtual work argument is used to prescribe nonconservative generalized forces on the right-hand side of Lagrange's equations.[4] In addition, if $\mathbf{F}_A$ is a conservative force with a potential energy function

$$U = U(\mathbf{x}_A) \, , \tag{27}$$

then

$$\mathbf{F}_A = -\frac{\partial U}{\partial \mathbf{x}_A}. \tag{28}$$

With the help of the identity

$$\mathbf{v}_A = \dot{\mathbf{x}}_A = \sum_{K=1}^{6} \dot{q}^K \frac{\partial \mathbf{x}_A}{\partial q^K}, \tag{29}$$

it is straightforward to show using the chain rule that

$$\begin{aligned} \mathbf{F}_A \cdot \frac{\partial \mathbf{v}_A}{\partial \dot{q}^K} = \mathbf{F}_A \cdot \frac{\partial \mathbf{x}_A}{\partial q^K} \\ = -\frac{\partial U}{\partial \mathbf{x}_A} \cdot \frac{\partial \mathbf{x}_A}{\partial q^K} \\ = -\frac{\partial U}{\partial q^K}. \end{aligned} \tag{30}$$

[4] See, for example, the lucid discussion in Baruh (1999, Chapter 4, Section 9).

Finally, Eqn. (26) allows one to easily write down Lagrange's equations in cases where a nonconservative follower force or a dynamic Coulomb friction force acts on the mechanical system. The most celebrated instance of the former case is Ziegler's pendulum.

3.2 Ideal Integrable Constraints

For the second simplification to the right-hand side of Lagrange's equations of motion, suppose that an integrable constraint is imposed on the system:

$$q^6 - f(t) = 0. \tag{31}$$

Then, if the constraint force $\mathbf{F}_c$ and constraint moment $\mathbf{M}_c$ satisfy Lagrange's prescription, it can be shown that[5]

$$\mathbf{F}_c \cdot \frac{\partial \bar{\mathbf{v}}}{\partial \dot{q}^K} + \mathbf{M}_c \cdot \frac{\partial \boldsymbol{\omega}}{\partial \dot{q}^K} = \mu \delta_6^K, \tag{32}$$

where μ is a scalar function (Lagrange multiplier) and δ_6^K is the Kronecker delta. This remarkable result enables Lagrange's equations of motion to decouple into two sets: one for the unconstrained (or generalized) coordinates and the other for the function μ.

In most treatments of Lagrange's equations of motion interest is restricted to the former set and the equations of motion for the generalized coordinates are formulated. For multibody systems involving rigid bodies connected by pin joints the savings in algebraic computations when attention is restricted to the generalized coordinates can be considerable. For instance, for the planar double pendulum, the number of equations of motion for the generalized coordinates is two while the remaining ten equations give expressions for the reaction forces and reaction moments at the two pin joints.

3.3 Potential Energies

The third and final simplification we wish to mention occurs if a conservative force $\mathbf{F}_{\text{con}}$ acting at the center of mass and a conservative moment (relative to the center of mass) $\mathbf{M}_{\text{con}}$ act on the rigid body. The combined mechanical power of these quantities is assumed to be equal to the negative rate of change of a potential energy function:

$$\mathbf{F}_{\text{con}} \cdot \bar{\mathbf{v}} + \mathbf{M}_{\text{con}} \cdot \boldsymbol{\omega} = -\dot{U}, \tag{33}$$

[5]This constraint in this case is sometimes known as ideal because $\mathbf{F}_c$ and $\mathbf{M}_c$ have no frictional components. For further details on constraint forces and constraint moments, see the expository paper by O'Reilly and Srinivasa (2014).

where

$$U = U(\bar{\mathbf{x}}, \mathbf{Q}), \qquad \dot{U} = \sum_{K=1}^{6} \frac{\partial U}{\partial q^K} \dot{q}^K. \tag{34}$$

Assuming that $\mathbf{F}_{\text{con}}$ and $\mathbf{M}_{\text{con}}$ are independent of the rates $\dot{q}^1, \ldots, \dot{q}^6$, it follows that

$$\mathbf{F}_{\text{con}} \cdot \frac{\partial \bar{\mathbf{v}}}{\partial \dot{q}^K} + \mathbf{M}_{\text{con}} \cdot \frac{\partial \boldsymbol{\omega}}{\partial \dot{q}^K} = -\frac{\partial U}{\partial q^K}. \tag{35}$$

Thus, we find that the partial derivatives of U with respect to the coordinates q^K are linear combinations of the components of the conservative force and conservative moment. Expressions for $\mathbf{F}_{\text{con}}$ and $\mathbf{M}_{\text{con}}$ can be established as gradients of U with respect to $\bar{\mathbf{x}}$ and $\mathbf{Q}$, respectively. We shall examine one such representation for $\mathbf{M}_{\text{con}}$ shortly.

3.4 A Canonical Form, Equilibria, and Linearization

Lagrange's equations of motion reveals a canonical form of the equations of motion. To elaborate, consider a system with N degrees-of-freedom whose kinetic energy can be expressed as a quadratic form and whose potential energy function depends on the N generalized coordinates:

$$T = \frac{1}{2} \sum_{I=1}^{N} \sum_{K=1}^{N} a_{IK} \dot{q}^I \dot{q}^K, \qquad U = U\left(q^1, \ldots, q^N\right), \tag{36}$$

where $a_{IK} = a_{IK}\left(q^1, \ldots, q^N\right)$. Such systems are pervasive in mechanics.

For such a system, it is known that[6]

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^K}\right) - \frac{\partial T}{\partial q^K} = \sum_{I=1}^{N} a_{KI} \ddot{q}^I + \sum_{S=1}^{N} \sum_{J=1}^{N} [SJ, K]\, \dot{q}^S \dot{q}^J. \tag{37}$$

Here, we have used the Christoffel symbols of the first kind $[SJ, K]$ to collect the quadratic velocity terms:

$$[SJ, K] = \frac{1}{2}\left(\frac{\partial a_{KJ}}{\partial q^S} + \frac{\partial a_{KS}}{\partial q^J} - \frac{\partial a_{SJ}}{\partial q^K}\right), \qquad (J, K, S = 1, \ldots, N). \tag{38}$$

If we assume that the only generalized forces acting on the system are conservative, then the equations of motion can be expressed as

$$\sum_{I=1}^{N} a_{KI} \ddot{q}^I + \sum_{S=1}^{N} \sum_{J=1}^{N} [SJ, K]\, \dot{q}^S \dot{q}^J = -\frac{\partial U}{\partial q^K}, \qquad (K = 1, \ldots, N). \tag{39}$$

[6] See, for example, the classic text by McConnell (1947).

An equilibrium of these equations satisfies the following $2N$ conditions:

$$\dot{q}^K = 0, \qquad q^K = q_0^K. \tag{40}$$

Examining the equations of motion (39), we observe that at an equilibrium the potential energy is extremized:

$$\frac{\partial U}{\partial q^K}\left(q_0^1, \ldots, q_0^N\right) = 0. \tag{41}$$

To establish the linearized equations of motion in the neighborhood of an equilibrium, we consider the following asymptotic expansion:

$$q^1 = q_0^1 + \epsilon\eta_1, \qquad \ldots, \qquad q^N = q_0^N + \epsilon\eta_N. \tag{42}$$

After substituting into (39), performing Taylor series expansions of U, a_{IK}, and $[SJ, K]$, using the equilibrium conditions, and ignoring terms of order ϵ^2 and higher, we find the following equations governing the linearized dynamics in a neighborhood of the equilibrium:

$$\mathsf{M}_0\ddot{\boldsymbol{\eta}} + \mathsf{K}_0\boldsymbol{\eta} = \mathbf{0}. \tag{43}$$

The mass matrix M_0 and stiffness matrix K_0 are both symmetric:

$$\boldsymbol{\eta} = \left[\begin{array}{c} \eta_1, \cdots, \eta_N \end{array}\right]^T, \tag{44}$$

$$\mathsf{M}_0 = \left[\begin{array}{ccc} a_{11}\left(q_0^1, \ldots q_0^N\right) & \cdots & a_{1N}\left(q_0^1, \ldots q_0^N\right) \\ \vdots & \ddots & \vdots \\ a_{1N}\left(q_0^1, \ldots q_0^N\right) & \cdots & a_{NN}\left(q_0^1, \ldots q_0^N\right) \end{array}\right], \tag{45}$$

$$\mathsf{K}_0 = \left[\begin{array}{ccc} \frac{\partial^2 U}{\partial q^1 \partial q^1}\left(q_0^1, \ldots q_0^N\right) & \cdots & \frac{\partial^2 U}{\partial q^1 \partial q^N}\left(q_0^1, \ldots q_0^N\right) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 U}{\partial q^1 \partial q^N}\left(q_0^1, \ldots q_0^N\right) & \cdots & \frac{\partial^2 U}{\partial q^N \partial q^N}\left(q_0^1, \ldots q_0^N\right) \end{array}\right]. \tag{46}$$

Thus, for many mechanical systems, Lagrange's equations of motion allows us to infer the equilibria of the system and the equations governing the linearized dynamics by simply computing the kinetic and potential energies and the derivatives of the latter energy.

4 Simple Conservative Moments

As remarked by Ziegler (1968, Page 30), the simplest nonconservative moment is a constant moment. He showed that a constant moment was nonconservative by examining the work done in rotating a rigid body through

180°. Such a motion can be accomplished in two equivalent manners. The first method is direct while the second involves successively rotating the body through 180° about two perpendicular axis. The work done by the moment in the latter case is zero and in the former case is non-zero. Whence, the constant moment is not conservative.

Ziegler's conclusion is surprising and involves an ingenious use of the Hamilton-Rodrigues theorem on finite rotations. His work also begs the question that if a constant moment isn't conservative, then which moment is? This question was also posed by Simmonds and answered by Antman (1972) and later by Simmonds (1984) himself. Antman's solution uses the Euler representation for a rotation featuring an axis of rotation and an angle of rotation. A simpler answer can be found using Euler angles.[7]

4.1 A Simple Representation for a Conservative Moment

Consider a potential energy U that depends solely on the orientation of the rigid body. Thus, we can express U as a function of the Euler angles and

$$\dot{U} = \frac{\partial U}{\partial \psi}\dot{\psi} + \frac{\partial U}{\partial \omega}\dot{\omega} + \frac{\partial U}{\partial \varphi}\dot{\varphi}. \tag{47}$$

However, using the dual Euler basis vectors,

$$\boldsymbol{\omega} \cdot \mathbf{g}^1 = \dot{\psi}, \qquad \boldsymbol{\omega} \cdot \mathbf{g}^2 = \dot{\omega}, \qquad \boldsymbol{\omega} \cdot \mathbf{g}^3 = \dot{\varphi}. \tag{48}$$

Substituting into the expression for $\dot{U}$,

$$\dot{U} = \left(\frac{\partial U}{\partial \psi}\mathbf{g}^1 + \frac{\partial U}{\partial \omega}\mathbf{g}^2 + \frac{\partial U}{\partial \varphi}\mathbf{g}^3 \right) \cdot \boldsymbol{\omega}. \tag{49}$$

Paralleling the case of a conservative force, we define a conservative moment $\mathbf{M}_{\text{con}}$ by postulating that

$$\dot{U} = -\mathbf{M}_{\text{con}} \cdot \boldsymbol{\omega}. \tag{50}$$

Assuming in addition that $\mathbf{M}_{\text{con}}$ is independent of $\boldsymbol{\omega}$, we find the representation

$$\mathbf{M}_{\text{con}} = -\frac{\partial U}{\partial \psi}\mathbf{g}^1 - \frac{\partial U}{\partial \omega}\mathbf{g}^2 - \frac{\partial U}{\partial \varphi}\mathbf{g}^3. \tag{51}$$

This is the simplest known representation of a conservative moment.[8] It has evident parallels to a representation of a conservative force acting on a particle that features a gradient expressed using contravariant basis vectors.

[7] Our developments and discussion are based on the works O'Reilly and Srinivasa (2002) and O'Reilly (2007, 2008).

[8] A compilation of representations for the gradient of U for various representations of the rotation $\mathbf{Q}$ can be found in O'Reilly (2008, Section 6.10).

4.2 Ziegler's Example Revisited

Returning to Ziegler's example, suppose that $M_0\mathbf{E}_3$ (where M_0 is a constant) is conservative. Then, the potential energy function associated with this moment would have to satisfy the following set of partial differential equations:

$$
\begin{aligned}
-\frac{\partial U}{\partial \psi} &= M_0\mathbf{E}_3 \cdot \mathbf{g}_1 = M_0\mathbf{E}_3 \cdot \mathbf{E}_3 = M_0, \\
-\frac{\partial U}{\partial \omega} &= M_0\mathbf{E}_3 \cdot \mathbf{g}_2 = M_0\mathbf{E}_3 \cdot \mathbf{e}_1^* = 0, \\
-\frac{\partial U}{\partial \varphi} &= M_0\mathbf{E}_3 \cdot \mathbf{g}_3 = M_0\mathbf{E}_3 \cdot \mathbf{e}_3 = M_0 \cos(\omega) . \qquad (52)
\end{aligned}
$$

However, the statements $\frac{\partial U}{\partial \omega} = 0$ and $\frac{\partial U}{\partial \varphi} = -M_0 \cos(\omega)$ are contradictory and we conclude that no such U can exist. Thus, $M_0\mathbf{E}_3$ is nonconservative.

4.3 Torsional Springs

We can use the representation for a conservative moment to establish an expression for the moment provided by a torsional spring. Suppose that the torsional spring's potential energy function is

$$U_{\text{spring}} = \frac{K}{2}(\psi - \psi_0)^2 , \qquad (53)$$

where ψ_0 is a constant and K is the spring constant. Invoking Eqn. (51), we find that

$$
\begin{aligned}
\mathbf{M}_{\text{spring}} &= -K(\psi - \psi_0)\,\mathbf{g}^1 \\
&= -K(\psi - \psi_0)\left(\mathbf{E}_3 - \cot(\omega)\left(-\cos(\psi)\,\mathbf{E}_2 - \sin(\psi)\,\mathbf{E}_1\right)\right) . \qquad (54)
\end{aligned}
$$

Observe that the spring moment has components orthogonal to the $\mathbf{E}_3$ direction. The fact that these components become unbounded as $\omega \to 0, \pi$ is a manifestation of the singularity in the 3-1-3 set of Euler angles at these values of ω. Unless the rigid body is constrained, it is necessary to switch to a complementary set of Euler angles such as the 3-2-1 set as ω approaches these singular values.

5 The Case of a Fixed Axis of Rotation

The singular behavior of the torsional spring moment in the previous section begs the question of how to deal with the case when the axis of rotation is constrained to be fixed. In this situation, choosing $\mathbf{E}_3$, say, to be parallel to the axis of rotation, we find that the angular velocity vector has the representation $\boldsymbol{\omega} = \dot{\psi}\mathbf{E}_3$. In addition, the motion of the rigid body is subject to a pair of constraints that can be expressed as

$$\boldsymbol{\omega} \cdot \mathbf{E}_1 = 0, \qquad \boldsymbol{\omega} \cdot \mathbf{E}_2 = 0. \tag{55}$$

A constraint moment with two independent components is needed to enforce these constraints:

$$\mathbf{M}_c = \mu_1 \mathbf{E}_1 + \mu_2 \mathbf{E}_2. \tag{56}$$

The moment $\mathbf{M}_c$ does no work and is nonconservative. If a pin joint is used to ensure that the axis of rotation stays constant, then the constraint moment $\mathbf{M}_c$ can be considered as a reaction moment provided by the pin joint. If the body is sliding on a flat surface, then $\mathbf{M}_c$ is the resultant moment provided by the normal forces acting on the surface of the body contacting the plane.

The potential energy of a torsional spring in this case is again given by

$$U_{\text{spring}} = \frac{K}{2} \left(\psi - \psi_0\right)^2. \tag{57}$$

Now, however, we seek solutions $\mathbf{M}_{\text{spring}}$ to the equation $\dot{U} = -\mathbf{M}_{\text{spring}} \cdot \boldsymbol{\omega}$ where $\boldsymbol{\omega} = \dot{\psi}\mathbf{E}_3$. The resulting solution is

$$\mathbf{M}_{\text{spring}} = -K \left(\psi - \psi_0\right) \mathbf{E}_3. \tag{58}$$

Fortunately, this expression has none of the issues associated with its three-dimensional counterpart (54). On a related note, as remarked in Ziegler (1968, Chapter 5), in the dynamics of rods where terminal moments of the form $M_0\mathbf{E}_3$ are applied, the boundary conditions often restrict the end rotation of the rod to be along $\mathbf{E}_3$. In this case, $M_0\mathbf{E}_3$ is a conservative moment. We refer the reader to O'Reilly (2017, Section 5.15) for further discussion of this case.

6 The Lagrange Top

To illustrate many of the previous developments, we now consider the classic example of an axisymmetric rigid body which is free to rotate about a point O. The rigid body is under the action of a vertical gravitational force. This mechanical system is known as the Lagrange top and its celebrated dynamics have a long and storied history.[9]

[9] For additional references and discussion, see Baruh (1999), Lewis et al. (1992), and Marsden and Ratiu (1999).

For the purposes of exposition, we establish the equation of motion for this system and include the effects of a constant moment $M_a\mathbf{E}_3$ acting on the body and a follower force $F_a\mathbf{e}_1$ acting at the tip of the top. Further, we allow the point O to be given a prescribed vertical motion $f(t)\mathbf{E}_3$. This motion can be imagined by assuming that the top is freely spinning about a point O and then the point O is oscillated in a vertical manner.

6.1 Kinematical Considerations

To establish the equations of motion for the top, we assign a set of 3-1-3 Euler angles to describe its orientation. The translational motion of the rigid body is characterized by a set of coordinates to describe the motion of O:

$$q^1 = \psi, \qquad q^2 = \omega, \qquad q^3 = \varphi,$$
$$q^4 = \mathbf{x}_O \cdot \mathbf{E}_1, \qquad q^5 = \mathbf{x}_O \cdot \mathbf{E}_2, \qquad q^6 = \mathbf{x}_O \cdot \mathbf{E}_3. \tag{59}$$

We assume that the position vector of the center of mass $\bar{X}$ relative to O is

$$\bar{\mathbf{x}} - \mathbf{x}_O = \ell\mathbf{e}_3. \tag{60}$$

The mass of the top is denoted by m and its inertia tensor is

$$\mathbf{J} = \lambda_t\left(\mathbf{I} - \mathbf{e}_3 \otimes \mathbf{e}_3\right) + \lambda_a\mathbf{e}_3 \otimes \mathbf{e}_3. \tag{61}$$

The velocity vector of the center of mass of the top has the representation

$$\bar{\mathbf{v}} = \dot{q}^4\mathbf{E}_1 + \dot{q}^5\mathbf{E}_2 + \dot{q}^6\mathbf{E}_3 + \omega_2\ell\mathbf{e}_1 - \omega_1\ell\mathbf{e}_2. \tag{62}$$

In this expression, the components $\omega_k = \boldsymbol{\omega} \cdot \mathbf{e}_k$ can be easily read off from Eqn. (11).

For future purposes, we note that

$$\frac{\partial\boldsymbol{\omega}}{\partial\dot{q}^k} = \mathbf{g}_k, \qquad \frac{\partial\boldsymbol{\omega}}{\partial\dot{q}^{k+3}} = \mathbf{0}, \qquad \frac{\partial\mathbf{v}_O}{\partial\dot{q}^k} = \mathbf{0}, \qquad \frac{\partial\mathbf{v}_O}{\partial\dot{q}^{k+3}} = \mathbf{E}_k, \tag{63}$$

where $k = 1, 2, 3$.

6.2 Constraints and Constraint Forces

The motion of the top is subject to three constraints. We have chosen the six coordinates in anticipation of these constraints being imposed. The three constraints can be expressed as follows:

$$q^4 = 0, \qquad q^5 = 0, \qquad q^6 - f(t) = 0. \tag{64}$$

These constraints can also be expressed in terms of a velocity vector:

$$\mathbf{v}_O \cdot \mathbf{E}_1 = 0, \qquad \mathbf{v}_O \cdot \mathbf{E}_2 = 0, \qquad \mathbf{v}_O \cdot \mathbf{E}_3 - \dot{f} = 0. \tag{65}$$

Referring to Section 3.2, this representation of the constraints allows us to easily appeal to Lagrange's prescription to write down a representation for the constraint force acting on the the top:

$$\mathbf{F}_c = \mu_1 \mathbf{E}_1 + \mu_2 \mathbf{E}_2 + \mu_3 \mathbf{E}_3 \text{ acting at } O. \tag{66}$$

This force is none other than the reaction force at O. With the help of the earlier results (63) we find that

$$\mathbf{F}_c \cdot \frac{\partial \mathbf{v}_O}{\partial \dot{q}^k} = 0, \qquad \mathbf{F}_c \cdot \frac{\partial \mathbf{v}_O}{\partial \dot{q}^{k+3}} = \mu_k. \tag{67}$$

Thus, as anticipated, the constraint force will be absent from three of the six Lagrange's equations of motion.

6.3 Kinetic and Potential Energies

The unconstrained kinetic energy of the top is

$$\begin{aligned} \hat{T} = \frac{m}{2} &\left(\sum_{i=1}^{3} \dot{q}^{3+i} \mathbf{E}_i + \omega_2 \ell \mathbf{e}_1 - \omega_1 \ell \mathbf{e}_2 \right) \cdot \left(\sum_{k=1}^{3} \dot{q}^{3+k} \mathbf{E}_k + \omega_2 \ell \mathbf{e}_1 - \omega_1 \ell \mathbf{e}_2 \right) \\ &+ \frac{\lambda_a}{2} \left(\dot{\varphi} + \dot{\psi} \cos(\omega) \right)^2 + \frac{\lambda_t}{2} \left(\dot{\omega}^2 + \dot{\psi}^2 \sin^2(\omega) \right). \end{aligned} \tag{68}$$

The unconstrained potential energy of the top is

$$\hat{U} = mg \mathbf{E}_3 \cdot (\mathbf{x}_O + \ell \mathbf{e}_3). \tag{69}$$

This potential energy is a function of q^6 and the Euler angle $q^2 = \omega$. We ornament T and U with hats $\hat{}$ to distinguish them from their constrained counterparts.

Imposing the constraints (65), the constrained kinetic and potential energies can be found:

$$\begin{aligned} T &= \frac{\lambda_a}{2} \left(\dot{\varphi} + \dot{\psi} \cos(\omega) \right)^2 + \frac{\lambda_t + m\ell^2}{2} \left(\dot{\omega}^2 + \dot{\psi}^2 \sin^2(\omega) \right) \\ &\quad + \frac{m}{2} \dot{f}^2 - m \dot{\omega} \dot{f} \ell \sin(\omega), \\ U &= mgf + mg\ell \cos(\omega). \end{aligned} \tag{70}$$

Observe that $\lambda_t^O = \lambda_t + m\ell^2$ is a mass moment of inertia relative to O.

6.4 The Equations of Motion

The rigid body is subject to an applied moment $M_a\mathbf{E}_3$ and a force $F_a\mathbf{e}_1$ which follows $\mathbf{e}_1$ and acts at the tip of the top. The latter point, which we denote by X_t is assumed to have a position vector $\ell_1\mathbf{e}_3$ relative to O. The equations governing the generalized coordinates ψ, ω, and φ are obtained from Lagrange's equation of motion:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^k}\right) - \frac{\partial T}{\partial q^k} = -\frac{\partial U}{\partial q^k} + M_a\mathbf{E}_3 \cdot \frac{\partial \boldsymbol{\omega}}{\partial \dot{q}^k} + F_a\mathbf{e}_1 \cdot \frac{\partial \mathbf{v}_t}{\partial \dot{q}^k}. \tag{71}$$

As $\mathbf{v}_t = \mathbf{v}_O + \boldsymbol{\omega} \times \ell_1\mathbf{e}_3$,

$$\frac{\partial \mathbf{v}_t}{\partial \dot{q}^k} = \ell_1\mathbf{g}_k \times \mathbf{e}_3, \tag{72}$$

where $k = 1, 2, 3$. Whence, the right-hand side of Lagrange's equations can be simplified:

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\psi}}\right) - \frac{\partial T}{\partial \psi} &= -\frac{\partial U}{\partial \psi} + \left(M_a + \ell_1 F_a \sin(\omega)\cos(\varphi)\right), \\
\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\omega}}\right) - \frac{\partial T}{\partial \omega} &= -\frac{\partial U}{\partial \omega} - \ell_1 F_a \sin(\varphi), \\
\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\varphi}}\right) - \frac{\partial T}{\partial \varphi} &= -\frac{\partial U}{\partial \varphi} + M_a \cos(\omega).
\end{aligned} \tag{73}$$

Notice that the nonconservative follower force and nonconservative moment both contribute terms that are coordinate dependent to the right-hand sides of Lagrange's equations of motion.

Evaluating the derivatives of T and U, we find that

$$\underbrace{\begin{bmatrix} \lambda_a\cos^2(\omega) + \lambda_t^O\sin^2(\omega) & 0 & \lambda_a\cos(\omega) \\ 0 & \lambda_t^O & 0 \\ \lambda_a\cos(\omega) & 0 & \lambda_a \end{bmatrix}}_{\mathsf{M}} \begin{bmatrix} \ddot{\psi} \\ \ddot{\omega} \\ \ddot{\varphi} \end{bmatrix} + \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}, \tag{74}$$

where

$$\begin{aligned}
\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} &= \begin{bmatrix} 2\left(\lambda_t^O - \lambda_a\right)\dot{\omega}\dot{\psi}\cos(\omega)\sin(\omega) - \lambda_a\dot{\varphi}\dot{\omega}\sin(\omega) \\ \lambda_a\dot{\varphi}\dot{\psi}\sin(\omega) - \left(\lambda_t^O - \lambda_a\right)\dot{\psi}^2\cos(\omega)\sin(\omega) \\ -\lambda_a\dot{\psi}\dot{\omega}\sin(\omega) \end{bmatrix}, \\
\begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} &= \begin{bmatrix} M_a + \ell_1 F_a \sin(\omega)\cos(\varphi) \\ m\left(g + \ddot{f}\right)\ell\sin(\omega) - \ell_1 F_a\sin(\varphi) \\ M_a\cos(\omega) \end{bmatrix}.
\end{aligned} \tag{75}$$

The mass matrix M on the left-hand side of Eqn. (74) can be inferred from the expression for the kinetic energy function $(70)_1$. The $\alpha_{1,2,3}$ terms are related to sums of Christoffel symbols featuring the derivatives of the components of M with respect to the coordinates.

We observe from the equations of motion that the effect of the vertical, time-varying motion of the point O is equivalent to changing the gravitational constant from g to $g + \ddot{f}$.

6.5 Equilibria and Linearized Equations of Motion

Suppose that the point O is fixed. A static equilibrium of the top corresponds to stationary values of ψ, ω, and φ. The static values, which are distinguished by a subscript 0, are

$$\begin{aligned} M_a + \ell_1 F_a \sin(\omega_0)\cos(\varphi_0) &= 0, \\ mg\ell \sin(\omega_0) - \ell_1 F_a \sin(\varphi_0) &= 0, \\ M_a \cos(\omega_0) &= 0. \end{aligned} \tag{76}$$

Whence, ψ_0 is arbitrary, and F_a and M_a must satisfy the latter pair of the following relations for a static equilibrium to exist:

$$\omega_0 = \frac{\pi}{2}, \qquad \sin(\varphi_0) = \frac{mg\ell}{F_a \ell_1}, \qquad \cos(\varphi_0) = -\frac{M_a}{F_a \ell_1}. \tag{77}$$

Thus, an infinite number of equilibrium states exist. As can be seen in Figure 5, the top is tilted at 90° to the vertical, ψ_0 is arbitrary, and $\tan(\varphi_0) = -\frac{mg\ell}{M_a}$.

We now consider perturbations from an equilibrium position:

$$\psi = \psi_0 + \epsilon\eta_1, \qquad \omega = \omega_0 + \epsilon\eta_2, \qquad \varphi = \varphi_0 + \epsilon\eta_3. \tag{78}$$

Inserting these expressions into the equations of motion (74), using the equilibrium conditions (77), performing Taylor series expansions, and ignoring terms of order ϵ^2, we find the linearized equations

$$\underbrace{\begin{bmatrix} \lambda_t^O & 0 & 0 \\ 0 & \lambda_t^O & 0 \\ 0 & 0 & \lambda_a \end{bmatrix}}_{\mathsf{M}_0} \begin{bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \\ \ddot{\eta}_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 & mg\ell \\ 0 & 0 & -M_a \\ 0 & M_a & 0 \end{bmatrix}}_{\mathsf{K}_0} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{79}$$

The linearized system has six eigenvalues:

$$0, \qquad 0, \qquad \pm\sqrt{\pm\sqrt{-\frac{M_a^2}{\lambda_a \lambda_t^O}}}. \tag{80}$$

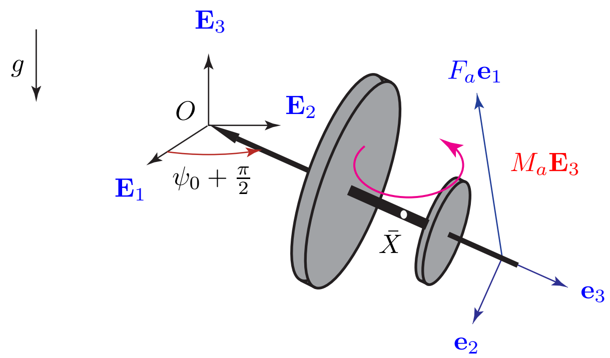

Figure 5: Equilibrium configuration of the Lagrange top subject to a follower force $F_a\mathbf{e}_1$ and a nonconservative moment $M_a\mathbf{E}_3$.

The pair of zero eigenvalues are a reflection of the arbitrariness of ψ_0. For non-zero values of M_a, the remaining four eigenvalues form a quartet. As two of the quartet have positive real parts, we conclude that the equilibrium is unstable.

Observe that the mass matrix M_0 is symmetric and can be readily deduced from the mass matrix M associated with the nonlinear equations of motion (74). In contrast to the conservative case presented in Eqn. (43), the stiffness matrix K_0 is asymmetric. This asymmetry can be attributed to the follower force load $F_a\mathbf{e}_1$ and the nonconservative moment $M_a\mathbf{E}_3$ acting on the top.

The asymmetry of the stiffness matrix in the linearized equations (79) is a generic feature of equilibria of nonconservative mechanical systems including models for brake squeal.[10] The asymmetry of the stiffness matrix also makes the equilibrium susceptible to dissipation-induced destabilization.[11]

[10] For additional details on brake squeal, the reader is referred to the review article Kinkaid et al. (2003).

[11] For a pseudospectral perspective on this topic see Kessler et al. (2007).

6.6 Solving for the Reaction Force

The constraint force $\mathbf{F}_c$ acting on the top can be determined from the equations of motion

$$\frac{d}{dt}\left(\frac{\partial \hat{T}}{\partial \dot{q}^{k+3}}\right) - \frac{\partial \hat{T}}{\partial q^{k+3}} = -\frac{\partial \hat{U}}{\partial q^{k+3}} + M_a \mathbf{E}_3 \cdot \frac{\partial \boldsymbol{\omega}}{\partial \dot{q}^{k+3}} + F_a \mathbf{e}_1 \cdot \frac{\partial \mathbf{v}_t}{\partial \dot{q}^{k+3}} + \mathbf{F}_c \cdot \frac{\partial \mathbf{v}_O}{\partial \dot{q}^{k+3}}. \tag{81}$$

These equations are computed using the unconstrained kinetic and potential energy functions. Using the identity $\mathbf{v}_t = \mathbf{v}_O + \boldsymbol{\omega} \times \ell_1 \mathbf{e}_3$ and Eqns. (63) and (67), we find the following expressions for the components of $\mathbf{F}_c = \sum_{k=1}^{3} \mu_k \mathbf{E}_k$:

$$\begin{aligned}
\mu_1 &= \frac{d}{dt}\left(\frac{\partial \hat{T}}{\partial \dot{q}^4}\right) - \frac{\partial \hat{T}}{\partial q^4} + \frac{\partial \hat{U}}{\partial q^4} - F_a \mathbf{e}_1 \cdot \mathbf{E}_1, \\
\mu_2 &= \frac{d}{dt}\left(\frac{\partial \hat{T}}{\partial \dot{q}^5}\right) - \frac{\partial \hat{T}}{\partial q^5} + \frac{\partial \hat{U}}{\partial q^5} - F_a \mathbf{e}_1 \cdot \mathbf{E}_2, \\
\mu_3 &= \frac{d}{dt}\left(\frac{\partial \hat{T}}{\partial \dot{q}^6}\right) - \frac{\partial \hat{T}}{\partial q^6} + \frac{\partial \hat{U}}{\partial q^6} - F_a \mathbf{e}_1 \cdot \mathbf{E}_3.
\end{aligned} \tag{82}$$

In other words, these three Lagrange's equations of motion are simply the projections of $\mathbf{F} = m\dot{\mathbf{v}}$ onto the Cartesian basis vectors.

7 The Satellite Dynamics Problem

We now return to the problem considered by Lagrange. Suppose a rigid body of mass m is orbiting a fixed spherically symmetric rigid body of mass M (cf. Figure 4(b)). We locate the origin of our coordinate system at the center of the body of mass M. Mac Cullagh's approximate expressions for the gravitational force $\mathbf{F}_n$, moment $\mathbf{M}_n$, and potential energy U_n on the rigid body of mass m are

$$\begin{aligned}
\mathbf{F}_n &\approx m\mathbf{g}, \\
\mathbf{M}_n &\approx \left(\frac{3GM}{R^3}\right) \mathbf{e}_R \times (\mathbf{J}\mathbf{e}_R), \\
U_n &\approx -\frac{GMm}{R} - \left(\frac{GM}{2R^3}\right) \mathrm{tr}(\mathbf{J}) + \left(\frac{3GM}{2R^3}\right) (\mathbf{e}_R \cdot (\mathbf{J}\mathbf{e}_R)),
\end{aligned} \tag{83}$$

where

$$m\mathbf{g} = -\frac{GMm}{R^2}\mathbf{e}_R - \frac{3GM}{2R^4}\left(2\mathbf{J} + \left(\text{tr}(\mathbf{J}) - 5\mathbf{e}_R \cdot \mathbf{J}\mathbf{e}_R\right)\mathbf{I}\right)\mathbf{e}_R, \tag{84}$$

and

$$R = ||\bar{\mathbf{x}}||\,, \qquad \mathbf{e}_R = \frac{\bar{\mathbf{x}}}{||\bar{\mathbf{x}}||}. \tag{85}$$

Notice that the unit vector $\mathbf{e}_R$ points from the center of mass of the spherically symmetric body to the center of mass of the body of mass m. In addition, if the moment of rotational inertia of the rigid body is ignored, then the moment vanishes and these expressions reduce to the familiar expression for a gravitational force on a particle of mass m. The moment $\mathbf{M}_n$ is often known as a gravity gradient torque in the satellite dynamics literature and features in studies of the precession of the equinoxes and librations of the Moon.[12]

Suppose a set of cylindrical polar coordinates (r, ϑ, z) are used to parameterize the position vector $\bar{\mathbf{x}}$ and a set of 3-1-3 Euler angles are used to parameterize $\mathbf{Q}$:

$$q^1 = r, \quad q^2 = \vartheta, \quad q^3 = z, \quad q^4 = \psi, \quad q^5 = \omega, \quad q^6 = \varphi. \tag{86}$$

That is

$$\bar{\mathbf{x}} = r\mathbf{e}_r + z\mathbf{E}_3, \qquad \bar{\mathbf{v}} = \dot{r}\mathbf{e}_r + r\dot{\vartheta}\mathbf{e}_\vartheta + \dot{z}\mathbf{E}_3, \tag{87}$$

where

$$\mathbf{e}_r = \cos(\vartheta)\,\mathbf{E}_1 + \sin(\vartheta)\,\mathbf{E}_2, \qquad \mathbf{e}_\vartheta = \cos(\vartheta)\,\mathbf{E}_2 - \sin(\vartheta)\,\mathbf{E}_1. \tag{88}$$

Then, with the help of Eqn. (35), we can readily identify the terms on the right-hand side of Lagrange's equations of motion (1) with the conservative force and conservative moment acting on the rigid body of mass m:

$$\begin{aligned} -\frac{\partial U_n}{\partial r} &= \mathbf{F}_n \cdot \mathbf{e}_r, & -\frac{\partial U_n}{\partial \vartheta} &= \mathbf{F}_n \cdot r\mathbf{e}_\vartheta, & -\frac{\partial U_n}{\partial z} &= \mathbf{F}_n \cdot \mathbf{E}_3, \\ -\frac{\partial U_n}{\partial \psi} &= \mathbf{M}_n \cdot \mathbf{g}_1, & -\frac{\partial U_n}{\partial \omega} &= \mathbf{M}_n \cdot \mathbf{g}_2, & -\frac{\partial U_n}{\partial \varphi} &= \mathbf{M}_n \cdot \mathbf{g}_3. \end{aligned} \tag{89}$$

Explicit expressions for U_n, $\mathbf{F}_n$, and $\mathbf{M}_n$ in terms of the Euler angles and the cylindrical polar coordinates can be readily obtained but they are very lengthy and disguise the important relations (89). We note that

$$\begin{aligned} \mathbf{F}_n &= -\frac{\partial U_n}{\partial r}\mathbf{e}_r - \frac{1}{r}\frac{\partial U_n}{\partial \vartheta}\mathbf{e}_\vartheta - \frac{\partial U_n}{\partial z}\mathbf{E}_3, \\ \mathbf{M}_n &= -\frac{\partial U_n}{\partial \psi}\mathbf{g}^1 - \frac{\partial U_n}{\partial \omega}\mathbf{g}^2 - \frac{\partial U_n}{\partial \varphi}\mathbf{g}^3. \end{aligned} \tag{90}$$

[12]See, e.g., Goldstein (1980, Section 5-8), Hughes (1986), or Kane et al. (1983).

These relations follow readily from Eqns. (15) and (89). Thus, we have been able to present transparent representations for the forces and moments featuring in Lagrange's equations (1).

After substituting for the Euler angles and the cylindrical polar coordinates into the expression for U_n, one finds the well-known result that U_n can be expressed as a function of $\psi + \vartheta$. With the help of Eqn. $(89)_{2,4}$, we can conclude that $\mathbf{F}_n \cdot r\mathbf{e}_\vartheta = \mathbf{M}_n \cdot \mathbf{E}_3$. As is often assumed in examining the dynamics of artificial and natural satellites, if the body is axisymmetric with $\lambda_1 = \lambda_2$ then one finds that U_n is independent of φ. Using Eqn. $(89)_6$, we deduce that $\mathbf{M}_n$ has no component along the axis of symmetry $\mathbf{e}_3$ and lies entirely in the plane spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$. Additionally, we can then conclude from Lagrange's equations of motion that the angular momentum component $\mathbf{H} \cdot \mathbf{e}_3$ is conserved.

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