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On Coordinate Singularities and Gimbal Lock in the Dynamics of Systems of Particles and Rigid Bodies

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Abstract Coordinate singularities and gimbal lock are two phenomena that feature in models for the dynamics of mechanical systems. The coordinates used to describe the motion of these systems induce a basis for the tangent space to the configuration manifold of the system. In this paper, we demonstrate how coordinate singularities manifest themselves as the failure of the induced basis to span the tangent space. Furthermore, we show that gimbal lock can be attributed to the system of applied forces and moments being normal to the configuration manifold. Consequently, the applied forces and moments have no effect on the equations of motion of the mechanical system at gimbal lock. Our treatment of gimbal lock and coordinate singularities clearly distinguishes these two phenomena from one another. We illustrate the treatment with a series of examples including a particle on a rotating plate, a bead in a rotating gimbal, and a rigid platform mounted in a Cardan suspension. These examples also serve to illuminate the effect of unilateral constraints on configuration manifolds, constraint impulses, and phase portraits.

Keywords Gimbal Lock · Coordinate Singularities · Configuration Manifold

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1 Introduction

The most popular notion of gimbal lock probably comes from NASA's Apollo program. Inside the inertial measurement unit of the Apollo spacecraft was a stable (inertial) platform about which the spacecraft rotated. The platform, upon which three orthogonal gyroscopes were mounted, was suspended in three gimbals to give it three degrees of freedom. The gyroscopes detected disturbances in the angular velocity of the platform. These disturbances were then quickly counteracted by the application of control torques from motors mounted at the gimbal bearings. At a particular orientation of the spacecraft, the three gimbals became coplanar and the control torques became ineffective at stabilizing the platform. Moreover, if the spacecraft then rotated about an axis normal to the common plane of the gimbals, the platform became "locked" and engaged in the same rotation as the spacecraft, thereby making the platform useless as an inertial reference. To correct for the locking of the gimbals, the spacecraft would have to pitch away from the problematic orientation and the platform would also need to be reoriented relative to the stars. To avoid gimbal lock entirely, four gimbals could be used with a control scheme to keep at least three suspension axes 90° apart. For the Apollo program, the engineers, who were fully aware of gimbal lock, opted to use three gimbals to save on weight costs, and preferred instead that the astronauts navigate around gimbal lock[1]. As discussed by Baruh [2, Chapter 7, Section 8] in the context of transmitting rotary motion between two shafts through a Cardan joint, gimbal lock arises as the inability to transmit motion when the shafts are perpendicular to one another.

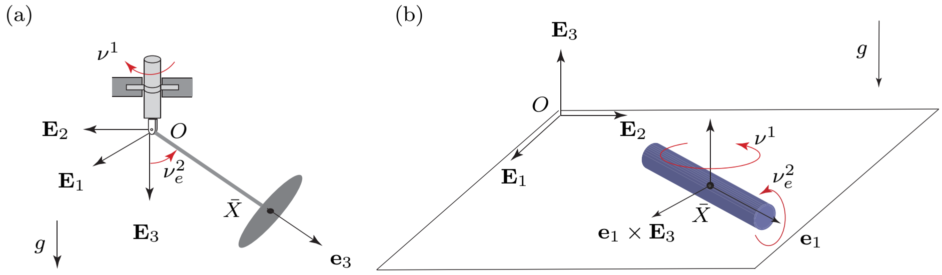

Fig. 1: *Examples of constrained rigid bodies. In (a), the configuration manifold of a pin-jointed whirling rigid body is a two-torus T^2, and, in (b) the configuration manifold of a cylinder sliding on a stationary surface is a four-dimensional manifold $\mathcal{E}^2 \times T^2$.*

Another phenomenon in the dynamics of rigid bodies, which is often (but not always) confused with gimbal lock, is the singularity that occurs in the Euler angle parameterizations of rotations. Related singularities arise when cylindrical polar coordinates and spherical polar coordinates are used to parameterize the motion of a mechanical system. What is not fully appreciated is that often these singularities are removed when the system is constrained. For instance, the motion of a rigid body

[1] For a comprehensive reference on gyroscopic devices mechanically suspended in gimbals, see [1]. For a technical overview of the inertial platforms used in the Apollo spacecraft missions, see [11] and [19]. Other useful references on inertial navigation include [7], [13], and [20, Chapter 6].

freely rotating about a fixed point can be parameterized by a set of Euler angles and the second of these angles has a limited range of 180°. However, when the rotation of the body is constrained (as in the examples shown in Figure 1), the configuration manifold of the system changes and the restriction on the range of the second angle needs to be removed so that two angles suffice to parameterize the rotation of the rigid body. In addition, the singularity in the parameterization of the motion vanishes. We shall discuss several other examples of this situation in the forthcoming pages.

In the present paper, we exploit recent work by Casey [3,4,5,6] on Lagrange's equations of motion for systems of particles and rigid bodies subject to systems of holonomic constraints. In Casey's work, properties of the mechanical system are used to construct a dynamic model of a single representative particle that moves on a configuration manifold of the system. That manifold is embedded in a high-dimensional space. We use the induced covariant basis for the tangent space to a point on the configuration manifold to clearly define coordinate singularities. In addition, we explain gimbal lock as an example of a phenomenon, distinct from a coordinate singularity, that arises when a system of applied forces and moments is equivalent to a generalized force vector that is orthogonal to the configuration manifold. The orthogonality of the generalized force vector implies that it has no effect on the equations of motion for the generalized coordinates.

As reviewed in Section 2, the configuration manifold of the mechanical system depends intimately on the inertial properties of the rigid bodies and particles that the system is composed of. We present three systems of increasing complexity that respectively feature cylindrical polar coordinates, spherical polar coordinates, and Euler angles to illustrate this point. The first system is a particle on a rotating plate discussed in Section 4. This is followed by a discussion of the dynamics of a particle on a spinning gimbal in Section 5. Our final, and most cogent, example is of a rigid body in a Cardan suspension (cf. Section 6). For instance, the configuration manifold of a rigid body suspended by massless gimbals is the real projective three-space $\mathbb{R}P^3$ while the configuration space of the same system if the inertia of the gimbals is included is the three-torus T^3. In particular, all of the examples discussed in this paper serve to illuminate an interesting phenomenon whereby coordinate singularities may be eliminated by the introduction of constraints and inertias. Our discussion of the Cardan suspension explicitly demonstrates how gimbal lock and orthogonality of the generalized force are intimately related.

2 Background

The systems of interest in this paper consist of particles and rigid bodies subject to integrable (or holonomic) constraints. We suppose there are p particles, R rigid bodies, and a set of H independent, integrable bilateral constraints. To establish the equations of motion for these systems, one chooses a set of $3p + 6R$ curvilinear coordinates, $\left\{q^1, \ldots, q^{3p+6R}\right\}$, such that the set of H constraints can be conveniently expressed as follows:

$$q^{3p+6R-K+1} - f^K(t) = 0, \qquad (K = 1, ..., H) . \tag{1}$$

If f^K is explicitly a function of time, then the constraint (1) is called time-dependent (or rheonomic). Otherwise, $\dot{f}^K = 0$, and constraint (1) is called time-independent (or scleronomic). The coordinates of interest can include Euler angles, Cartesian coordinates, and spherical polar coordinates, among others. In the sequel, we adopt the following compact notation:

$$\mathsf{q}^* = \left[q^1, \ldots, q^{3p+6R}\right]^T, \qquad \dot{\mathsf{q}}^* = \left[\dot{q}^1, \ldots, \dot{q}^{3p+6R}\right]^T, \tag{2}$$

where $[\cdot]^T$ denotes the transpose. To avoid confusion where it might arise, we distinguish those quantities where the integrable constraints have not been imposed with an asterisk $*$.

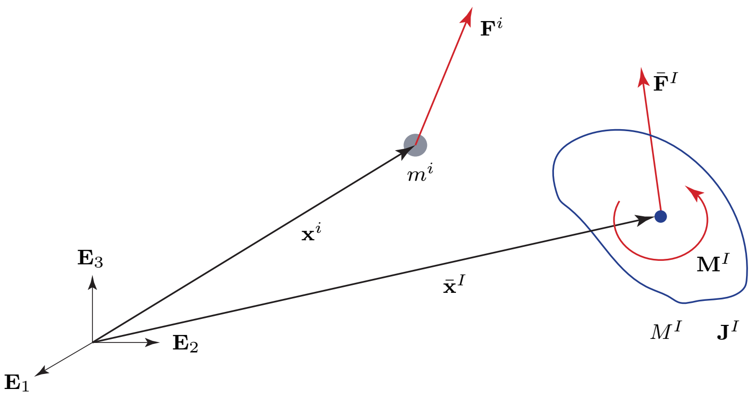

Fig. 2: Schematic of the ith particle and the Ith rigid body for the system of interest. A resultant force $\mathbf{F}^i$ acts on the particle of mass m^i while a resultant force $\bar{\mathbf{F}}^I$ and a resultant moment about the center of mass $\mathbf{M}^I$ acts on the Ith rigid body of mass M^I and moment of inertia tensor about the center of mass $\mathbf{J}^I$. The orthonormal basis vectors $\mathbf{E}_1, \mathbf{E}_2$, and $\mathbf{E}_3$ form an orthonormal basis for the physical space $\mathbb{E}^3$.

As illustrated in Figure 2, we denote the position vector of the ith particle by $\mathbf{x}^i$ where $i = 1, \ldots, p$, the position vector of the center of mass of the Ith rigid body by $\bar{\mathbf{x}}^I$, and the rotation tensor of the Ith rigid body that takes the reference configuration into the current configuration by $\mathbf{Q}^I$ where $I = 1, \ldots, R$.[2] For the system in the absence of constraints, we can describe the position vector of each particle, the position vector of the center of mass of each rigid body, and the rotation tensor of each rigid body as a function of the $3p + 6R$ coordinates:

$$\begin{aligned} \mathbf{x}^i &= \mathbf{x}^i\left(\mathsf{q}^*\right), \qquad (i = 1, \ldots, p), \\ \bar{\mathbf{x}}^I = \bar{\mathbf{x}}^I\left(\mathsf{q}^*\right), &\qquad \mathbf{Q}^I = \mathbf{Q}^I\left(\mathsf{q}^*\right), \qquad (I = 1, \ldots, R). \end{aligned} \tag{3}$$

The unconstrained kinetic energy T^* of the system can be expressed as the sum of the kinetic energies of the individual particles and rigid bodies:

$$T^* = \sum_{i=1}^{p} \frac{1}{2} m^i \mathbf{v}^i \cdot \mathbf{v}^i + \sum_{I=1}^{R} \left(\frac{1}{2} M^I \bar{\mathbf{v}}^I \cdot \bar{\mathbf{v}}^I + \frac{1}{2} \boldsymbol{\omega}^I \cdot \mathbf{J}^I \boldsymbol{\omega}^I\right). \tag{4}$$

[2] Additional background for a variety of sources on rotation tensors, Euler angles, Euler bases, and dual Euler bases are collected in Appendix A.

Here, $\mathbf{v}^i = \dot{\mathbf{x}}^i$ is the velocity vector of the ith particle, m^i is the mass of the ith particle, $\bar{\mathbf{v}}^I = \dot{\bar{\mathbf{x}}}^I$ is the velocity vector of the center of mass of the Ith rigid body, M^I is the mass of the Ith rigid body, $\boldsymbol{\omega}^I$ is the angular velocity vector of the Ith rigid body, and $\mathbf{J}^I$ is the moment of inertia tensor about the center of mass of the Ith rigid body. This energy can be expressed as a function of the $3p + 6R$ coordinates and their time derivatives as

$$T^* = \frac{1}{2} \left(\dot{\mathsf{q}}^*\right)^T \mathsf{M}^* \dot{\mathsf{q}}^*, \tag{5}$$

where the components of the $(3p+6R) \times (3p+6R)$ symmetric matrix M^* may depend on the coordinates q^*.

When integrable constraints are imposed, the kinetic energy of the system reduces to an explicit function of the generalized coordinates, $q^1, \ldots, q^N$, generalized velocities, $\dot{q}^1, \ldots, \dot{q}^N$, and time, t. We index the constrained coordinates as $q^{N+1}, \ldots, q^{3p+6R}$. Here, N, the number of degrees of freedom of the system, is

$$N = 3p + 6R - H. \tag{6}$$

We introduce the abbreviated notations

$$\mathsf{q} = \left[q^1, \ldots, q^N\right]^T, \qquad \dot{\mathsf{q}} = \left[\dot{q}^1, \ldots, \dot{q}^N\right]^T. \tag{7}$$

The kinetic energy T of the constrained system has a well-known decomposition into a term T_2 that is quadratic in the generalized velocities, a term T_1 that is linear in the generalized velocities, and a term T_0 that is independent of the generalized velocities:

$$T = T_2 \left(\dot{\mathsf{q}}, \mathsf{q}, t\right) + T_1 \left(\dot{\mathsf{q}}, \mathsf{q}, t\right) + T_0 \left(\mathsf{q}, t\right). \tag{8}$$

T_1 and T_0 are nonzero in the presence of time-dependent constraints. The quadratic term has a decomposition

$$T_2 = \frac{1}{2} \dot{\mathsf{q}}^T \mathsf{M} \dot{\mathsf{q}}, \tag{9}$$

where the $N \times N$-dimensional matrix M is known as the mass matrix. If all of the constraints are time-independent, then this matrix will not be an explicit function of time. The function T and matrix M can be obtained from T^* and M^*, respectively, by imposing the system of integrable constraints (1).

We next assume that a resultant force $\mathbf{F}^i$ acts on the ith particle, a resultant force $\bar{\mathbf{F}}^I$ acts at the center of mass of the Ith rigid body, and a resultant moment $\mathbf{M}^I$ acts about the center of mass of the Ith rigid body. Then, Lagrange's equations of motion for the system are

$$\frac{d}{dt} \left(\frac{\partial T^*}{\partial \dot{q}^J} \right) - \frac{\partial T^*}{\partial q^J} = Q_J, \qquad (J = 1, \ldots, 3p + 6R). \tag{10}$$

The generalized force Q_J has the decomposition

$$Q_J = \sum_{i=1}^{p} \mathbf{F}^i \cdot \frac{\partial \mathbf{v}^i}{\partial \dot{q}^J} + \sum_{I=1}^{R} \left(\bar{\mathbf{F}}^I \cdot \frac{\partial \bar{\mathbf{v}}^I}{\partial \dot{q}^J} + \mathbf{M}^I \cdot \frac{\partial \boldsymbol{\omega}^I}{\partial \dot{q}^J} \right). \tag{11}$$

That is, Q_J is a linear combination of the resultant forces and moments acting on the constituents of the system.

The equations of motion for the integrably constrained system can often be simply found from a subset of (10). First, we assume that the integrable constraints are ideal. That is, the constraint forces and moments that enforce the constraints satisfy Lagrange's prescription.[3] Examples of this case include a particle that is free to move on a smooth curve or surface and a top that is mounted on a smooth ball-and-socket joint. For the ideal case where the choice of coordinates induces a constraint of the form (1), it can be proven (see, for example, [16]) that the constraint forces and moments acting on the system do not contribute to a subset of the Q_Js:

$$\sum_{i=1}^{p} \mathbf{F}_c^i \cdot \frac{\partial \mathbf{v}^i}{\partial \dot{q}^A} + \sum_{I=1}^{R} \left(\bar{\mathbf{F}}_c^I \cdot \frac{\partial \bar{\mathbf{v}}^I}{\partial \dot{q}^A} + \mathbf{M}_c^I \cdot \frac{\partial \boldsymbol{\omega}^I}{\partial \dot{q}^A} \right) = 0, \qquad (A = 1, \ldots, N) . \qquad (12)$$

Here, $\mathbf{F}_c^i$ is the constraint force acting on the ith particle, $\bar{\mathbf{F}}_c^I$ is the constraint force acting at the center of mass of the Ith rigid body, and $\mathbf{M}_c^I$ is the constraint moment acting at the center of mass of the Ith rigid body. In this case, Lagrange's equations of motion for the N generalized coordinates, $q^1, \ldots, q^N$, can be computed using the constrained kinetic energy (8):

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^A}\right) - \frac{\partial T}{\partial q^A} = Q_A, \qquad (A = 1, \ldots, N) , \qquad (13)$$

and Q_A is composed solely of applied forces and moments. A canonical form of (13) can be found by substituting for T and expanding the partial derivatives:

$$\mathsf{M}\ddot{\mathsf{q}} + \mathsf{f}\left(\mathsf{q}, \dot{\mathsf{q}}, t\right) = \mathsf{Q}, \qquad (14)$$

where $\mathsf{Q} = [Q_1, \ldots, Q_N]^T$ and f is an N-dimensional column array. In classic texts, where the constraints are assumed to be time-independent, it is shown how the components of f can be expressed in terms of products of Christoffel symbols and generalized velocities.

Geometric interpretations of (10) and (13) can be found in the papers of Casey [3, 4, 5, 6]. Using these works, one can construct a single representative particle of mass m moving in a $(3p + 12R)$-dimensional vector space, $\mathcal{C}^{3p+12R}$. An arbitrary point in this space represents a motion of a system of p particles and R continua. By assuming rigidity of the continua in the system, configurations are limited to a $(3p+6R)$-dimensional subset of this space, $\mathcal{M}^*$, the configuration manifold of the unconstrained system. The position vector $\mathbf{r}$ of the representative particle and its resultant force $\boldsymbol{\Phi}$ are such that the equations of motion for the particle $m\ddot{\mathbf{r}} = \boldsymbol{\Phi}$ are equivalent to the collective balance laws for the particles and rigid bodies. In addition, the kinetic energy of the unconstrained particle is equivalent to kinetic energy T^* of the unconstrained system: $\frac{1}{2}m\left[\dot{\mathbf{r}}, \dot{\mathbf{r}}\right] = T^*$, where $[\cdot, \cdot]$ is an inner product on $\mathcal{C}^{3p+12R}$ (see [4, 6]). A single integrable constraint restricts the representative particle's motion to be in a $(3p+6R-1)$-dimensional hypersurface. When H independent integrable constraints are imposed on the system as in equation (1), the particle is constrained to move on the N-dimensional configuration manifold $\mathcal{M}$ (cf. Figure 3) that is formed as the intersection of the H hypersurfaces. The resulting constraint force is denoted by $\boldsymbol{\Phi}_c$, which represents the collective forces and moments that manifest themselves in the presence of constraints (i.e., normal forces, reaction forces and moments, tension forces, etc.).

[3] In other words, the generalized constraint forces do no virtual work in any motion of the system compatible with the constraints.

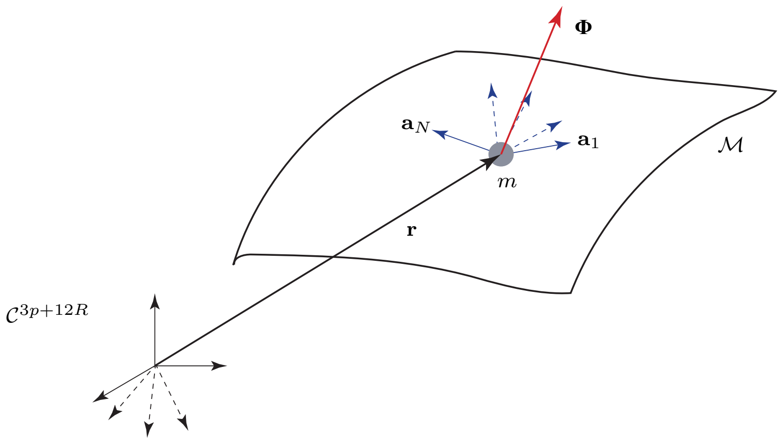

Fig. 3: Schematic of the representative particle of mass m moving on an N-dimensional configuration manifold $\mathcal{M}$. This manifold is a subset of a $(3p + 12R)$-dimensional Euclidean space $\mathcal{C}^{3p+12R}$.

The generalized coordinates $q^1, \ldots, q^N$ parameterize $\mathcal{M}$. The position vector of the representative particle in $\mathcal{C}^{3p+12R}$ locating a point on $\mathcal{M}^*$ is constructed as follows:

$$\mathbf{r} = \left(\mathbf{x}^1, \ldots, \mathbf{x}^p, \bar{\mathbf{x}}^1, \ldots, \bar{\mathbf{x}}^R, \mathbf{Q}^1, \ldots, \mathbf{Q}^R\right). \tag{15}$$

The coordinates $q^1, \ldots, q^{3p+6R}$ are then used to define a set of covariant and contravariant basis vectors in a standard manner:

$$\mathbf{a}_J = \frac{\partial \mathbf{r}}{\partial q^J}, \qquad \mathbf{a}^J = \nabla q^J, \qquad (J = 1, \ldots, 3p + 6R). \tag{16}$$

The subset $\{\mathbf{a}_1, \ldots \mathbf{a}_N\}$ of the covariant basis vectors span the tangent space, $T_P\mathcal{M}$ at the point $P \in \mathcal{M}$ where P indicates the point occupied by the representative particle. Most notably,

$$\begin{aligned} Q_J &= \boldsymbol{\Phi} \cdot \mathbf{a}_J \\ &= \sum_{i=1}^{p} \mathbf{F}^i \cdot \frac{\partial \mathbf{v}^i}{\partial \dot{q}^J} + \sum_{I=1}^{R} \left(\bar{\mathbf{F}}^I \cdot \frac{\partial \bar{\mathbf{v}}^I}{\partial \dot{q}^J} + \mathbf{M}^I \cdot \frac{\partial \boldsymbol{\omega}^I}{\partial \dot{q}^J} \right). \end{aligned} \tag{17}$$

With the help of (12), the vanishing of the constraint forces and moments from the right-hand side of Lagrange's equations (10) is equivalent to the constraint force $\boldsymbol{\Phi}_c$ acting on the particle of mass m being normal to $\mathcal{M}$: $\boldsymbol{\Phi}_c \cdot \mathbf{a}_A = 0$ for $A = 1, \ldots, N$.

For a mechanical system consisting of a single particle, the representative particle and the single particle are identical. In addition to this case, in the sequel, systems of a rigid body and a particle and systems of multiple rigid bodies are considered. Along with these examples, some readers may find the detailed discussion of the construction of the representative particle for systems of particles in the textbook [15] to be of interest.

3 Coordinate Singularities and Orthogonality of Loadings

The coordinate system $\left\{q^1, \ldots, q^N\right\}$ for the configuration manifold will in some cases be prone to singularities. These singularities generally manifest themselves as ambiguities in the values of $\left\{q^1, \ldots, q^N\right\}$ at some point(s) of $\mathcal{M}$. For the systems of interest in the subsequent examples, we say that a coordinate system has a singularity when the set $\{\mathbf{a}_1, \ldots \mathbf{a}_N\}$ fails to be linearly independent (and thus a basis for $T_P\mathcal{M}$) at some point $P \in \mathcal{M}$.

We note that our definition of a coordinate singularity subsumes Greenwood's definition in [8, Chapter 7, Section 13] and the notion of Euler angle singularities given in Shuster [17, Page 460]. While the kinetic energy T^* is a positive-definite quadratic function of velocity vectors $\mathbf{v}^i$, $\boldsymbol{\omega}^I$, and $\bar{\mathbf{v}}^I$, the constrained kinetic energy expressed as a function of the generalized coordinates q and their velocities $\dot{\mathsf{q}}$ may not be a positive-definite function of $\dot{\mathsf{q}}$. Specifically, at a coordinate singularity, the mass matrix M becomes singular and the kinetic energy T_2 ceases to be a positive-definite function of the velocities $\dot{\mathsf{q}}$. As M approaches the singularity, the equations of motion (14) become stiff and ill-posed, resulting in time integration issues with numerical schemes.

One of the main aims of this paper is to elaborate on the connections and distinctions between singularities and the notion of gimbal lock. To help achieve this goal, we now define the notion of an *orthogonal system of applied forces and moments*. Here, orthogonality pertains to the configuration manifold. We define a system of applied forces and moments such that $\mathbf{F}^i_a$ is the applied force on the ith particle, $\bar{\mathbf{F}}^I_a$ is the applied force acting at the center of mass of the Ith rigid body, and $\mathbf{M}^I_a$ is the applied moment relative to the center of mass of the Ith rigid body. The loading is said to be orthogonal to $\mathcal{M}$ if

$$\sum_{i=1}^{P} \mathbf{F}^i_a \cdot \frac{\partial \mathbf{v}^i}{\partial \dot{q}^A} + \sum_{I=1}^{R} \left(\bar{\mathbf{F}}^I_a \cdot \frac{\partial \bar{\mathbf{v}}^I}{\partial \dot{q}^A} + \mathbf{M}^I_a \cdot \frac{\partial \boldsymbol{\omega}^I}{\partial \dot{q}^A} \right) = 0, \qquad (A = 1, \ldots, N)\,. \tag{18}$$

Observe that this condition is equivalent to the corresponding force vector, which we denote by $\boldsymbol{\Phi}_a$, acting on the representative particle of mass m as having vanishing covariant components: $\boldsymbol{\Phi}_a \cdot \mathbf{a}_A = 0$ for $A = 1, \ldots, N$. Our definition of an orthogonal system of applied forces and moments presumes that there is no coordinate singularity in the bases at the relevant point on the configuration manifold.

After inspecting the right-hand side of (13), it should be noted that such a system of forces and moments will have no effect on the equations of motion (13) for the generalized coordinates of the system. If the constraints on a system are ideal and integrable, then the system of constraint forces and moments acting on the system satisfy (18). In the sequel, we shall find that gimbal lock is an example of an instance where an applied loading is orthogonal to the configuration manifold and thus has no effect on the dynamics of the generalized coordinates.

4 A Particle sliding on a Plate

Our first example features the motion of a particle that can be described using a cylindrical polar coordinate system. After first investigating the dynamics of a particle that is free to move in space and whose motion is parameterized by cylindrical

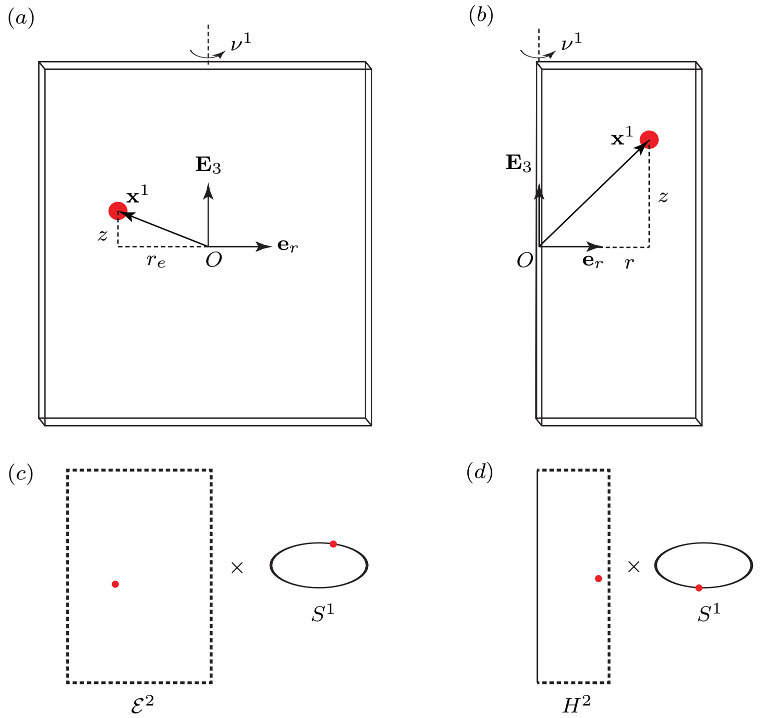

Fig. 4: *A particle sliding on a (a) full plate and a (b) half-plate and their respective configuration manifolds, (c)* $\mathcal{E}^2 \times S^1$ *and (d)* $H^2 \times S^1$*. The imposition of a unilateral constraint for the half-plate results in a configuration manifold with boundary. It is also important to note that* $r \in [0, \infty)$ *whereas* $r_e \in (-\infty, \infty)$*.*

polar coordinates, we explore how the imposition of an integrable constraint changes the configuration manifold to a 2-dimensional space that can be parameterized in a singularity-free manner. Our discussion then continues with an examination of a single particle constrained to move in a plane and a closely related particle-plate system (see Figure 4). By incorporating the inertia of the plate into the model for the system, we find that the configuration manifold becomes $\mathcal{E}^2 \times S^1$. It is demonstrated how, for the particle in a plate, an applied force in a particular direction becomes orthogonal to $\mathcal{M}$ when the particle aligns with the rotation axis of the plate. This singular physical behavior may be taken as a rudimentary definition of gimbal lock and is an example of an orthogonal loading (cf. Eqn. (18)).

4.1 Singularity in the Cylindrical Polar Coordinate System and a Unconstrained Particle

A particle free to move in flat Euclidean 3-space, $\mathcal{E}^3$, has the same dynamics as the representative particle. Here, we have the following equivalences: $\mathcal{C}^3 = \mathbb{E}^3$, $\mathcal{M}^* = \mathcal{M} = \mathcal{E}^3$, $m = m^1$, $\mathbf{r} = \mathbf{x}^1$, $\mathbf{a}_i = \frac{\partial \mathbf{v}^1}{\partial \dot{q}^i}$, and $\mathbf{\Phi} = \mathbf{F}^1$. For clarity, we denote $\mathcal{E}^n$ as an n-dimensional Euclidean (flat) manifold while $\mathbb{E}^n$ denotes a Euclidean (equipped with inner product) vector space. The physical particle and representative particle both have the respective position and velocity vectors

$$\mathbf{r} = r\mathbf{e}_r + z\mathbf{E}_3, \qquad \mathbf{v} = \dot{r}\mathbf{e}_r + \dot{\theta}r\mathbf{e}_\theta + \dot{z}\mathbf{E}_3, \tag{19}$$

where $\{r, \theta, z\}$ are cylindrical polar coordinates for $\mathcal{E}^3$, $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ is a fixed orthonormal triad in $\mathbb{E}^3$, $\mathbf{e}_r = \cos(\theta)\,\mathbf{E}_1 + \sin(\theta)\,\mathbf{E}_2$, and $\mathbf{e}_\theta = \mathbf{E}_3 \times \mathbf{e}_r$. By restricting $r \in [0, \infty)$, $\theta \in [0, 2\pi)$, and $z \in (-\infty, \infty)$, the parametrization is one-to-one everywhere in $\mathcal{E}^3$ except at $r = 0$, where θ is arbitrary. The covariant basis for the tangent space of $\mathcal{E}^3$,

$$\mathbf{a}_1 = \mathbf{e}_r, \qquad \mathbf{a}_2 = r\mathbf{e}_\theta, \qquad \mathbf{a}_3 = \mathbf{E}_3, \tag{20}$$

is required to span $T_P\mathcal{E}^3 = \mathbb{E}^3$ for all $P \in \mathcal{E}^3$. The coordinate singularity manifests itself as a failure of the covariant basis to span $\mathbb{E}^3$ at $r = 0$. As a consequence, the set $\{\mathbf{a}_i\}$ is unable to capture a component of velocity in the $\mathbf{e}_\theta$ direction. The kinetic energy has the representation

$$T^* = \frac{1}{2}m\sum_{i=1}^{3}\sum_{j=1}^{3}\dot{q}^i\dot{q}^j\mathbf{a}_i\cdot\mathbf{a}_j = \frac{1}{2}m\left(\dot{r}^2 + \dot{\theta}^2r^2 + \dot{z}^2\right). \tag{21}$$

We see that T^* fails to be a positive-definite quadratic form at $r^2 = 0$ (and $\mathbf{a}_2\cdot\mathbf{a}_2 = 0$) where it is only positive-semidefinite. The equations of motion are

$$m\begin{bmatrix}1 & 0 & 0\\ 0 & r^2 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\ddot{r}\\ \ddot{\theta}\\ \ddot{z}\end{bmatrix} + m\begin{bmatrix}\dot{\theta}^2\,(-r)\\ 2\dot{r}\dot{\theta}\,(r)\\ 0\end{bmatrix} = \begin{bmatrix}\mathbf{\Phi}\cdot\mathbf{e}_r\\ \mathbf{\Phi}\cdot r\mathbf{e}_\theta\\ \mathbf{\Phi}\cdot\mathbf{E}_3\end{bmatrix}, \tag{22}$$

which is of the canonical form (14). When $r = 0$, the mass matrix, whose entries are $\mathsf{M}_{ij} = m\mathbf{a}_i\cdot\mathbf{a}_j$, becomes singular and therefore non-invertible. The non-invertibility of M presents an issue in the numerical integration of equation (22), where a value of $r = 0$ yields infinitely large values for the coordinate accelerations. We emphasize that for a force acting on the physical particle $\mathbf{F}^1$ in $\mathbf{e}_\theta$ at $r = 0$, the fact that $\mathbf{F}^1\cdot\frac{\partial\mathbf{v}^1}{\partial\dot{\theta}}$ vanishes does not correspond to $\mathbf{\Phi}$ being orthogonal to the configuration manifold. Rather, it is due to a singularity in the coordinate system.

4.2 Eliminating the Singularity by Imposing Constraints on the Particle

The coordinate singularity at $r = 0$ may be eliminated by explicitly defining a value for θ at every instant of time. The resulting constraint changes the configuration manifold $\mathcal{M}$ from a 3-dimensional to a 2-dimensional object. The single integrable constraint can be expressed in the equivalent forms (1):

$$\theta - f(t) = 0, \qquad \mathbf{v}\cdot\frac{1}{r}\mathbf{e}_\theta - \dot{f}(t) = 0. \tag{23}$$

Lagrange's prescription (see, e.g. [15,16]) yields the required constraint force as

$$\mathbf{\Phi}_c = \mathbf{F}_c^1 = \mu\frac{1}{r}\mathbf{e}_\theta. \tag{24}$$

If we choose to use the original cylindrical coordinates that we inherit from $\mathcal{E}^3$, we must also impose the unilateral constraint $r \geq 0$, which may be written as

$$\mathbf{v} \cdot \mathbf{e}_r \geq 0 \quad \text{at} \quad r = 0. \tag{25}$$

To enforce the unilateral constraint, a constraint impulse $\hat{\mathbf{F}}_c^1 = \hat{\mathbf{\Phi}}_c$ acts on the particle at $r = 0$ of the form

$$\hat{\mathbf{\Phi}}_c = [[m\mathbf{v}]] = m\mathbf{v}^+ - m\mathbf{v}^-, \tag{26}$$

where $\mathbf{v}^+$ and $\mathbf{v}^-$ are the velocities of the particle immediately before and after impact, respectively. The constraint impulse was not present for the case of the unconstrained particle despite requiring $r \geq 0$ since θ was allowed to jump as the particle passed through $r = 0$. In this example, $\mathcal{M} = H^2$, a half-plane rotating at angular frequency $\dot{f}$. The unilateral constraint has resulted in a configuration manifold with boundary. The constrained velocity vector is

$$\mathbf{v} = \dot{r}\mathbf{a}_1 + \dot{z}\mathbf{a}_2 + \frac{\partial \mathbf{r}}{\partial t} = \dot{r}\mathbf{e}_r + \dot{z}\mathbf{E}_3 + \dot{f}r\mathbf{e}_\theta, \tag{27}$$

where $\dot{f}r\mathbf{e}_\theta$ is the velocity of the point in H^2 that coincides with the particle. The covariant basis vectors are easily read off:

$$\mathbf{a}_1 = \frac{\partial \mathbf{v}^1}{\partial \dot{r}} = \mathbf{e}_r, \qquad \mathbf{a}_2 = \frac{\partial \mathbf{v}^1}{\partial \dot{z}} = \mathbf{E}_3. \tag{28}$$

The set $\{\mathbf{a}_1, \mathbf{a}_2\}$ provides a well-defined basis for describing $\mathbf{v} - \frac{\partial \mathbf{r}}{\partial t}$, the velocity relative to the configuration manifold that lies in the rotating tangent space $T_P H^2 = \mathbb{E}^2$ for all $P \in H^2$. The constrained kinetic energy is

$$T = T_2 + T_0 = \frac{1}{2}m\left(\dot{r}^2 + \dot{z}^2\right) + \frac{1}{2}m\dot{f}^2 r^2. \tag{29}$$

T_2 is a positive-definite quadratic form of $\dot{\mathsf{q}} = [\dot{r}, \dot{z}]^T$ for all q, even at $r = 0$. Owing to the fact that constraint (23) is of the form (1), we have:

$$\mathbf{F}_c^1 \cdot \frac{\partial \mathbf{v}^1}{\partial \dot{r}} = 0, \qquad \mathbf{F}_c^1 \cdot \frac{\partial \mathbf{v}^1}{\partial \dot{z}} = 0. \tag{30}$$

Thus, $\mathbf{\Phi}_c \cdot \mathbf{a}_1 = \mathbf{\Phi}_c \cdot \mathbf{a}_2 = 0$, and the constraint force does not appear in Q_1 or Q_2 thereby unaffecting the free motion of the particle in H^2. The force $\mathbf{\Phi}_c$ is orthogonal to $\mathcal{M}$ but is not powerless, since the imposed constraint is time-dependent. The equations of motion and the multiplier μ are easily computed from (22):

$$m\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\ddot{r}\\ \ddot{z}\end{bmatrix} + m\begin{bmatrix}\dot{f}^2(-r)\\ 0\end{bmatrix} = \begin{bmatrix}\mathbf{\Phi}_a \cdot \mathbf{e}_r\\ \mathbf{\Phi}_a \cdot \mathbf{E}_3\end{bmatrix}, \tag{31}$$

with the required constraint force $\mathbf{\Phi}_c = \mathbf{F}_c^1 = \mu\frac{1}{r}\mathbf{e}_\theta$ given by

$$\mathbf{\Phi}_c = \left(mr\ddot{f} + 2m\dot{r}\dot{f} - \mathbf{\Phi}_a \cdot \mathbf{e}_\theta\right)\mathbf{e}_\theta. \tag{32}$$

The mass matrix is now non-singular for all values of q and $\mathbf{\Phi}_c$ is non-singular, even at $r = 0$. A component of force in $\mathbf{e}_\theta$ at $r = 0$ is captured by the constraint force. The constraint impulse $\hat{\mathbf{\Phi}}_c$ is determined from the impulse-momentum form of the balance of linear momentum. Assumptions about the (in)elasticity of the impact are needed to solve $\hat{\mathbf{\Phi}}_c$.

If the physical particle is constrained to move in a rotating infinite plane, $\mathcal{E}^2$, then one must extend r by defining a new coordinate, $r_e \in (-\infty, \infty)$. On $\mathcal{E}^3$, the coordinates $\{r_e, \theta, z\}$ are two-to-one except for the usual ambiguity at $r_e = 0$. Once constraint (23) is imposed, $\{r_e, z\}$ give a one-to-one coordinatization of $\mathcal{E}^2$ everywhere. The equations of motion are identical to those of the particle in the half-plane:

$$m \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{r}_e \\ \ddot{z} \end{bmatrix} + m \begin{bmatrix} \dot{f}^2 \left(-r_e\right) \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi}_a \cdot \mathbf{e}_r \\ \mathbf{\Phi}_a \cdot \mathbf{E}_3 \end{bmatrix}, \tag{33}$$

with

$$m r_e^2 \ddot{f} + 2 m r_e \dot{r}_e \dot{f} - \mathbf{\Phi}_a \cdot r_e \mathbf{e}_\theta = \mu. \tag{34}$$

In this case, no impulse is required at $r_e = 0$. When numerically integrating equation (31), one must manually impose the assumption that $r \geq 0$, or else it will be assumed that r_e is desired. An integration of equation (31) without applying $\hat{\mathbf{\Phi}}_c$ is equivalent to integrating equation (33).

4.3 The Effects of the Inertia of a Supporting Plate

Consider a mechanical system consisting of a particle constrained to slide in a plate that is engaged in fixed-axis rotation about $\mathbf{E}_3$, as shown in Figure 4(a). Let $\{x_1, x_2, x_3\}$ be rectangular Cartesian coordinates for $\bar{\mathbf{x}}^1$ measured from the fixed point O, $\{r_e, \theta, z\}$ be extended cylindrical polar coordinates for $\mathbf{x}^1 - \bar{\mathbf{x}}^1$, and $\left\{\nu^1, \nu^2, \nu^3\right\}$ be a 3-2-1 set of Euler angles for $\mathbf{Q}^1$. The reference configuration of the plate is such that its center of mass coincides with O and a normal to its face is in the $\mathbf{E}_2$ direction. A corotational basis $\left\{\mathbf{e}_i^1\right\}$ is attached to the plate in its current configuration. For the unconstrained system, we have the following kinematical quantities:

$$\bar{\mathbf{x}}^1 = \sum_{i=1}^{3} x_i \mathbf{E}_i, \qquad \mathbf{x}^1 - \bar{\mathbf{x}}^1 = r_e \mathbf{e}_r + z \mathbf{e}_3^1, \tag{35}$$

and the components of $\mathbf{Q}^1$ are gotten from equation (110). Note that the set $\{\mathbf{e}_r, \mathbf{e}_\theta\}$ is relative to $\left\{\mathbf{e}_1^1, \mathbf{e}_2^1\right\}$. The position vector of the representative particle in $\mathcal{C}^{15}$ is

$$\mathbf{r} = \left(\mathbf{x}^1, \bar{\mathbf{x}}^1, \mathbf{Q}^1\right). \tag{36}$$

The unconstrained velocities in the physical system are

$$\bar{\mathbf{v}}^1 = \sum_{i=1}^{3} \dot{x}_i \mathbf{E}_i, \qquad \boldsymbol{\omega}^1 = \sum_{i=1}^{3} \dot{\nu}^i \mathbf{g}_i,$$
$$\mathbf{v}^1 - \bar{\mathbf{v}}^1 = \dot{r}_e \mathbf{e}_r + \dot{\theta} r_e \mathbf{e}_\theta + \dot{z} \mathbf{e}_3^1 + \boldsymbol{\omega}^1 \times \left(\mathbf{x}^1 - \bar{\mathbf{x}}^1\right), \tag{37}$$

The generalized coordinates for the system are $\left\{q^1, q^2, q^3\right\} = \left\{r_e, \nu^1, z\right\}$ while the constrained coordinates are $\left\{q^4, \ldots, q^9\right\} = \left\{x_1, x_2, x_3, \nu^2, \nu^3, \theta\right\}$. The constraints are all integrable and of the form (1):

$$x_1 = 0, \quad x_2 = 0, \quad x_3 = 0, \quad \nu^2 = 0, \quad \nu^3 = 0, \quad \theta = 0. \tag{38}$$

In terms of velocity vectors we have

$$\bar{\mathbf{v}}^1 \cdot \mathbf{E}_1 = 0, \qquad \bar{\mathbf{v}}^1 \cdot \mathbf{E}_2 = 0, \qquad \bar{\mathbf{v}}^1 \cdot \mathbf{E}_3 = 0, \qquad \boldsymbol{\omega}^1 \cdot \mathbf{e}_\theta = 0, \qquad \boldsymbol{\omega}^1 \cdot \mathbf{e}_r = 0,$$
$$\mathbf{v}^1 \cdot \mathbf{e}_\theta - \bar{\mathbf{v}}^1 \cdot \mathbf{e}_\theta - \boldsymbol{\omega}^1 \cdot \left(r_e \mathbf{e}_3^1 - z \mathbf{e}_r \right) = 0. \tag{39}$$

After the constraints are imposed, the configuration manifold becomes $\mathcal{E}^2 \times S^1$ (cf. Figure 4(c)). The constrained velocity vectors are

$$\mathbf{v}^1 = \dot{r}_e \mathbf{e}_r + \dot{\nu}^1 r_e \mathbf{e}_\theta + \dot{z} \mathbf{E}_3, \quad \bar{\mathbf{v}}^1 = \mathbf{0}, \quad \boldsymbol{\omega}^1 = \dot{\nu}^1 \mathbf{E}_3. \tag{40}$$

Lagrange's prescription yields the required constraint forces and moments as

$$\mathbf{F}_c^1 = \mu_6 \mathbf{e}_\theta, \qquad \bar{\mathbf{F}}_c^1 = \mu_1 \mathbf{E}_1 + \mu_2 \mathbf{E}_2 + \mu_3 \mathbf{E}_3 - \mu_6 \mathbf{e}_\theta,$$
$$\mathbf{M}_c^1 = \mu_4 \mathbf{e}_\theta + \mu_5 \mathbf{e}_r + \mu_6 \left(z \mathbf{e}_r - r_e \mathbf{E}_3 \right). \tag{41}$$

The sets $\left\{ \frac{\partial \mathbf{v}^1}{\partial \dot{q}^J} \right\}$, $\left\{ \frac{\partial \bar{\mathbf{v}}^1}{\partial \dot{q}^J} \right\}$, and $\left\{ \frac{\partial \boldsymbol{\omega}^1}{\partial \dot{q}^J} \right\}$ are calculated before the constraints are imposed. Then, after imposing the constraints, we obtain:

$$\left\{ \frac{\partial \mathbf{v}^1}{\partial \dot{q}^J} \right\} = \left\{ \mathbf{e}_r, r_e \mathbf{e}_\theta, \mathbf{E}_3, \mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3, -r_e \mathbf{E}_3 + z \mathbf{e}_r, -z \mathbf{e}_\theta, r_e \mathbf{e}_\theta \right\},$$
$$\left\{ \frac{\partial \bar{\mathbf{v}}^1}{\partial \dot{q}^J} \right\} = \left\{ \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3, \mathbf{0}, \mathbf{0}, \mathbf{0} \right\}, \qquad \left\{ \frac{\partial \boldsymbol{\omega}^1}{\partial \dot{q}^J} \right\} = \left\{ \mathbf{0}, \mathbf{E}_3, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{e}_\theta, \mathbf{e}_r, \mathbf{0} \right\}. \tag{42}$$

We have a covariant basis for the 3-dimensional tangent space $T_P \left(\mathcal{E}^2 \times S^1 \right)$:

$$\mathbf{a}_1 = \frac{\partial \mathbf{r}}{\partial r_e} = \left(\mathbf{e}_r, \mathbf{0}, \mathbf{O} \right), \qquad \mathbf{a}_2 = \frac{\partial \mathbf{r}}{\partial \nu^1} = \left(r_e \mathbf{e}_\theta, \mathbf{0}, \mathsf{skwt}\left(\mathbf{E}_3 \right) \mathbf{Q}^1 \right),$$
$$\mathbf{a}_3 = \frac{\partial \mathbf{r}}{\partial z} = \left(\mathbf{E}_3, \mathbf{0}, \mathbf{O} \right), \tag{43}$$

where $\mathbf{O}$ is the zero-tensor for the 9-dimensional vector space of linear second-order tensors. The set $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ constitutes an orthogonal basis for every tangent space of $\mathcal{E}^2 \times S^1$. Therefore, the coordinate system is singularity-free. As anticipated, the constraint forces and moments only contribute to the components of Q that are normal to the configuration manifold:

$$\mathbf{F}_c^1 \cdot \frac{\partial \mathbf{v}^1}{\partial \dot{q}^A} + \bar{\mathbf{F}}_c^1 \cdot \frac{\partial \bar{\mathbf{v}}^1}{\partial \dot{q}^A} + \mathbf{M}_c^1 \cdot \frac{\partial \boldsymbol{\omega}^1}{\partial \dot{q}^A} = 0, \quad (A = 1, 2, 3). \tag{44}$$

Thus, the constraint forces do not contribute to any Q_A in the equations of motion. Equivalently, $\boldsymbol{\Phi}_c$ is everywhere orthogonal to $\mathcal{E}^2 \times S^1$. We have simple expressions for Q_J, $J = 1, \ldots, 9$, as

$$Q_1 = \mathbf{F}_a^1 \cdot \mathbf{e}_r, \qquad Q_2 = \mathbf{F}_a^1 \cdot r_e \mathbf{e}_\theta + \mathbf{M}_a^1 \cdot \mathbf{E}_3, \qquad Q_3 = \mathbf{F}_a^1 \cdot \mathbf{E}_3,$$
$$Q_{(3+i)} = \mu_i + \mathbf{F}_a^1 \cdot \mathbf{E}_i + \bar{\mathbf{F}}_a^1 \cdot \mathbf{E}_i, \qquad (i = 1, 2, 3),$$
$$Q_7 = \mu_4 + \mathbf{F}_a^1 \cdot \left(-r_e \mathbf{E}_3 + z \mathbf{e}_r \right) + \mathbf{M}_a^1 \cdot \mathbf{e}_\theta,$$
$$Q_8 = \mu_5 + \mathbf{F}_a^1 \cdot -z \mathbf{e}_\theta + \mathbf{M}_a^1 \cdot \mathbf{e}_r, \qquad Q_9 = \mu_6 r_e + \mathbf{F}_a^1 \cdot r_e \mathbf{e}_\theta. \tag{45}$$

Let the plate possess a finite radius of gyration k about $\mathbf{E}_3$ through O. The constrained kinetic energy of the system (and the corresponding representative particle) is then

$$T = T_2 = \frac{1}{2}m^1\left(\dot{r}_e^2 + \dot{z}^2 + \left(\dot{\nu}^1\right)^2 r_e^2\right) + \frac{1}{2}M^1k^2\left(\dot{\nu}^1\right)^2, \tag{46}$$

which is non-degenerate for all $\mathsf{q} = [r_e, \nu^1, z]^T$. The equations of motion are

$$\begin{bmatrix} m^1 & 0 & 0 \\ 0 & m^1r_e^2 + M^1k^2 & 0 \\ 0 & 0 & m^1 \end{bmatrix}\begin{bmatrix} \ddot{r}_e \\ \ddot{\nu}^1 \\ \ddot{z} \end{bmatrix} + m^1\begin{bmatrix} \left(\dot{\nu}^1\right)^2(-r_e) \\ 2\dot{r}_e\dot{\nu}^1(r_e) \\ 0 \end{bmatrix} = \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix}. \tag{47}$$

Comparing these equations of motion to equation (22), the presence of the Mk^2 term makes the mass matrix (and the kinetic energy) in this example non-singular. If we think of Mk^2 as controlling the radius of S^1, then when $Mk^2 = 0$, the configuration manifold degenerates into $\mathcal{E}^3$, and $\left\{r_e, \nu^1, z\right\}$ become the usual extended cylindrical polar coordinates with singularities along the z-axis.

A force applied to the particle in $\mathbf{e}_\theta$ contributes to the generalized force Q_2 everywhere except where $r_e = 0$. The applied force gets appropriately accounted for in Q_1, Q_2, Q_8, and Q_9, as the corresponding $\mathbf{\Phi}_a$ is perfectly orthogonal to $\mathcal{E}^2 \times S^1$. The fact that $\frac{\partial \mathbf{v}^1}{\partial \dot{\nu}^1} = \mathbf{0}$ at $r_e = 0$ is not due to a coordinate singularity, but to the nature of the constrained particle velocity and its allowable directions of movement.

To describe a particle in a half-plate, as in Figure 4(b), we use the coordinate $r \in [0, \infty)$ and prescribe a constraint impulse to ensure that $\dot{r} \geq 0$ at $r = 0$. The resulting equations of motion are the same, with point O now locating a fixed point of the plate rather than its center of mass. The configuration manifold becomes $H^2 \times S^1$ (Figure 4(d)), a 3-manifold with boundary for which $\{r, \theta, z\}$ are non-degenerate.

5 A Particle Sliding on a Gimbal

In this section, we make analogous conclusions to the previous section by investigating singularities in the spherical polar coordinate system. In concordance with the previous section, we first impose constraints on the problematic coordinates to show how the singularities are avoided. Then, we mount a particle in a full gimbal and a half-gimbal and discuss the resulting effect on the configuration manifold when the gimbal's inertia is included. The particle in a half-gimbal is treated in [15, Exercise 3.7] while a discussion of a particle in a full gimbal is found in [9, Example 2.5], and [10, Chapter 2, Section 8]. The preceding references do not include the inertia of the gimbal, but its inclusion has some surprising consequences.

5.1 Spherical Polar Coordinate System and Particle Dynamics

As previously noted, the dynamics of a single particle free to move in space are equivalent to the dynamics for the representative particle in the abstract space. In spherical polar coordinates, the position and velocity vectors of both the physical particle and the representative particle are, respectively,

$$\mathbf{r} = R\mathbf{e}_R, \qquad \mathbf{v} = \dot{R}\mathbf{e}_R + \dot{\theta}R\sin(\phi)\,\mathbf{e}_\theta + \dot{\phi}R\mathbf{e}_\phi \tag{48}$$

where $\{R, \phi, \theta\}$ are spherical polar coordinates for $\mathcal{E}^3$, $\mathbf{e}_R = \cos(\phi)\,\mathbf{E}_3 + \sin(\phi)\,\mathbf{e}_r$, and $\mathbf{e}_\phi = \mathbf{e}_\theta \times \mathbf{e}_R$. By making the restrictions $R \in [0, \infty)$, $\theta \in [0, 2\pi)$, and $\phi \in [0, \pi]$, the parametrization is one-to-one everywhere except at $R = 0$, where θ and ϕ are arbitrary and at $\phi = 0, \pi$, where θ is arbitrary. The covariant basis is

$$\mathbf{a}_1 = \mathbf{e}_R, \qquad \mathbf{a}_2 = R\sin(\phi)\,\mathbf{e}_\theta, \qquad \mathbf{a}_3 = R\mathbf{e}_\phi. \tag{49}$$

The set $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ fails to be a basis at $R = 0$ and $\phi = 0, \pi$. The spherical polar coordinates therefore have singularities at these values. The kinetic energy has the representation

$$T^* = \frac{1}{2}m\left(\dot{R}^2 + \dot{\theta}^2 R^2 \sin^2(\phi) + \dot{\phi}^2 R^2\right). \tag{50}$$

Clearly, T^* fails to be a well-defined quadratic form at $R = 0$ and at $\phi = 0, \pi$. The equations of motion are

$$m\begin{bmatrix} 1 & 0 & 0 \\ 0 & R^2\sin^2(\phi) & 0 \\ 0 & 0 & R^2 \end{bmatrix}\begin{bmatrix} \ddot{R} \\ \ddot{\theta} \\ \ddot{\phi} \end{bmatrix} + m\begin{bmatrix} \dot{\theta}^2\left(-R\sin^2(\phi)\right) + \dot{\phi}^2(-R) \\ 2\dot{R}\dot{\theta}\left(R\sin^2(\phi)\right) + 2\dot{\theta}\dot{\phi}\left(R^2\sin(\phi)\cos(\phi)\right) \\ \dot{\theta}^2\left(-R^2\sin(\phi)\cos(\phi)\right) + 2\dot{R}\dot{\phi}(R) \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi}\cdot\mathbf{e}_R \\ \mathbf{\Phi}\cdot R\sin(\phi)\,\mathbf{e}_\theta \\ \mathbf{\Phi}\cdot R\mathbf{e}_\phi \end{bmatrix}. \tag{51}$$

When $R = 0$ or $\phi = 0, \pi$, the mass matrix is non-invertible, which is equivalent to the observed degeneracy in the kinetic energy. Also, the coordinates fail to capture components of a force in $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$ at the singularities.

5.2 A Particle on a Rotating Half-Line

Any potential ambiguities in θ or ϕ are avoided by explicitly prescribing their value for all time as follows:

$$\theta - f^1(t) = 0, \quad \phi - f^2(t) = 0. \tag{52}$$

Physically, the particle is constrained to move on a half-line issuing from the origin that rotates about the vertical with an angular speed $\dot{f}^1$ while rotating about $\mathbf{e}_\theta$ with a angular speed $\dot{f}^2$. The corresponding constraints on the velocity vector are

$$\mathbf{v}\cdot\frac{1}{R\sin(f^2)}\mathbf{e}_\theta - \dot{f}^1(t) = 0, \quad \mathbf{v}\cdot\frac{1}{R}\mathbf{e}_\phi - \dot{f}^2 = 0. \tag{53}$$

The resulting constrained velocity vector is

$$\mathbf{v} = \dot{R}\mathbf{e}_R + \dot{f}^1 R\sin\left(f^2\right)\mathbf{e}_\theta + \dot{f}^2 R\mathbf{e}_\phi. \tag{54}$$

If we choose to inherit R from the parametrization for $\mathcal{E}^3$, then the imposition of these constraints yields $\mathcal{M} = H^1$, a half-line or ray. We have $\mathbf{a}_1 = \mathbf{e}_R$ providing a well-defined basis for $T_P H^1 = \mathbb{E}^1$, so we expect no coordinate singularities, and the vector $\mathbf{v} - \frac{\partial \mathbf{r}}{\partial t} = \dot{R}\mathbf{e}_R$ lies in this space. Lagrange's prescription gives the required constraint force as

$$\mathbf{\Phi}_c = \mu_1\frac{1}{R\sin(f^2)}\mathbf{e}_\theta + \mu_2\frac{1}{R}\mathbf{e}_\phi. \tag{55}$$

Since R came from the ambient space, we must enforce the unilateral constraint $R \geq 0$, which is of the form

$$\mathbf{v} \cdot \mathbf{e}_R \geq 0 \quad \text{at} \quad R = 0. \tag{56}$$

The corresponding constraint impulse required at $R = 0$ has the prescription

$$\hat{\mathbf{\Phi}}_c = m\mathbf{v}^+ - m\mathbf{v}^-. \tag{57}$$

The constrained kinetic energy is

$$T = T_2 + T_0 = \frac{1}{2}m\dot{R}^2 + \frac{1}{2}m\left(\dot{f}^1\dot{f}^1 R^2 \sin^2\left(f^2\right) + \dot{f}^2\dot{f}^2 R^2\right) \tag{58}$$

The kinetic energy capturing movement relative to the configuration manifold, T_2, is clearly a non-degenerate quadratic form of $\dot{R}$. The sole equation of motion and constraint force is

$$m\ddot{R} - mR\left(\dot{f}^2\dot{f}^2 + \dot{f}^1\dot{f}^1\sin^2\left(f^2\right)\right) = \mathbf{\Phi}_a \cdot \mathbf{e}_R, \tag{59}$$

with

$$\begin{aligned}\mathbf{\Phi}_c = &\left(mR\ddot{f}^1\sin\left(f^2\right) + 2m\dot{f}^2\left(\dot{R}\sin\left(f^2\right) + R\dot{f}^2\cos\left(f^2\right)\right) - \mathbf{\Phi}_a \cdot \mathbf{e}_\theta\right)\mathbf{e}_\theta \\ &+ \left(mR\ddot{f}^2 + 2m\dot{R}\dot{f}^2 - mR\sin\left(f^2\right)\cos\left(f^2\right)\dot{f}^1\dot{f}^1 - \mathbf{\Phi}_a \cdot \mathbf{e}_\phi\right)\mathbf{e}_\phi.\end{aligned} \tag{60}$$

The mass matrix in equation (59) is simply m and is invertible as a nonzero scalar. To describe a particle on a full line with configuration manifold $\mathcal{E}^1$, an extension of R from the range $[0,\infty)$ to $R_e \in (-\infty,\infty)$ is required. The resulting equation of motion is identical to equation (59) with constraint force identical to equation (60) but no constraint impulse $\hat{\mathbf{\Phi}}_c$ is required.

5.3 Adding System Inertias to a Particle on a Gimbal

Consider a mechanical system consisting of a particle sliding on a gimbal of radius R_0 that is engaged in fixed-axis rotation about $\mathbf{E}_3$, as shown in Figure 5(b). Let $\{x_1, x_2, x_3\}$ be rectangular Cartesian coordinates for $\bar{\mathbf{x}}^1$, $\{R, \theta, \phi_e\}$ be spherical polar coordinates for $\mathbf{x}^1 - \bar{\mathbf{x}}^1$, and $\left\{\nu^1, \nu^2, \nu^3\right\}$ be a set of 3-2-1 Euler angles for $\mathbf{Q}^1$. Here, ϕ_e is the extension of the ordinary polar angle ϕ, so that $\phi_e \in [0, 2\pi)$. The reference configuration of the gimbal is such that a normal to its face is in the $\mathbf{E}_2$-direction. A corotational basis $\left\{\mathbf{e}_i^1\right\}$ is attached to the gimbal in its current configuration. For the unconstrained system, we have the following kinematical quantities:

$$\bar{\mathbf{x}}^1 = \sum_{i=1}^{3} x_i \mathbf{E}_i, \qquad \mathbf{x}^1 - \bar{\mathbf{x}}^1 = R\mathbf{e}_R, \tag{61}$$

and the components of $\mathbf{Q}^1$ are gotten from equation (110). Note that the set $\left\{\mathbf{e}_R, \mathbf{e}_\theta, \mathbf{e}_\phi\right\}$ is relative to the corotational basis $\left\{\mathbf{e}_i^1\right\}$. The position vector of the representative particle in $\mathcal{C}^{15}$ is

$$\mathbf{r} = \left(\mathbf{x}^1, \bar{\mathbf{x}}^1, \mathbf{Q}^1\right). \tag{62}$$

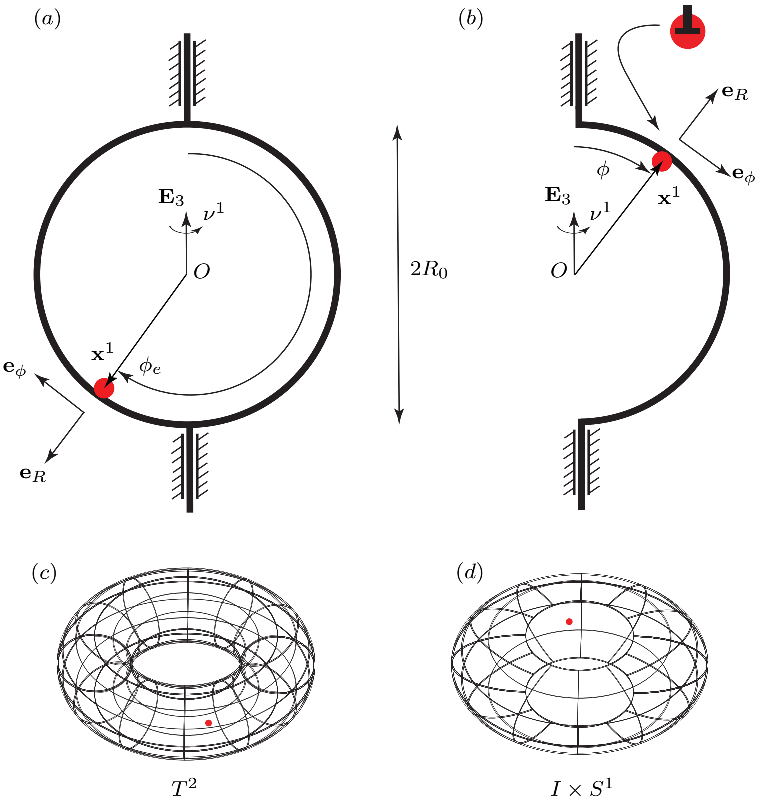

Fig. 5: *A particle sliding on a (a) full gimbal and (b) half-gimbal has and their respective configuration manifolds, (c)* T^2 *and (d)* $I \times S^1$*. The imposition of a unilateral constraint for the half-gimbal results in a configuration manifold with boundary.*

The unconstrained velocities in the physical system are

$$\bar{\mathbf{v}}^1 = \sum_{i=1}^{3} \dot{x}_i \mathbf{E}_i, \qquad \boldsymbol{\omega}^1 = \sum_{i=1}^{3} \dot{\nu}^i \mathbf{g}_i,$$
$$\mathbf{v}^1 - \bar{\mathbf{v}}^1 = \dot{R}\mathbf{e}_R + \dot{\theta} R \sin(\phi_e)\, \mathbf{e}_\theta + \dot{\phi}_e R \mathbf{e}_\phi + \boldsymbol{\omega}^1 \times \left(\mathbf{x}^1 - \bar{\mathbf{x}}^1\right). \tag{63}$$

The generalized coordinates for the system are $\left\{q^1, q^2\right\} = \left\{\nu^1, \phi_e\right\}$ while the constrained coordinates are $\left\{q^3, \ldots, q^9\right\} = \left\{x_1, x_2, x_3, \nu^2, \nu^3, R, \theta\right\}$. Before the constraints are applied, we have the following sets:

$$\left\{\frac{\partial \mathbf{v}^1}{\partial \dot{q}^J}\right\} = \left\{\mathbf{g}_1 \times \mathbf{d}, R\mathbf{e}_\phi, \mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3, \mathbf{g}_2 \times \mathbf{d}, \mathbf{g}_3 \times \mathbf{d}, \mathbf{e}_R, R\sin(\phi_e)\, \mathbf{e}_\theta\right\},$$
$$\left\{\frac{\partial \bar{\mathbf{v}}^1}{\partial \dot{q}^J}\right\} = \left\{\mathbf{0}, \mathbf{0}, \mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}\right\}, \qquad \left\{\frac{\partial \boldsymbol{\omega}^1}{\partial \dot{q}^J}\right\} = \left\{\mathbf{g}_1, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{g}_2, \mathbf{g}_3, \mathbf{0}, \mathbf{0}\right\}. \tag{64}$$

where $\mathbf{d} = \mathbf{x}^1 - \bar{\mathbf{x}}^1$. The seven integrable constraints are

$$x_1 = 0, \quad x_2 = 0, \quad x_3 = 0, \quad \nu^2 = 0, \quad \nu^3 = 0, \quad R - R_0 = 0, \quad \theta = 0. \tag{65}$$

In terms of velocity vectors, we have

$$\begin{gathered}\bar{\mathbf{v}}^1 \cdot \mathbf{E}_1 = 0, \quad \bar{\mathbf{v}}^1 \cdot \mathbf{E}_2 = 0, \quad \bar{\mathbf{v}}^1 \cdot \mathbf{E}_3 = 0, \quad \boldsymbol{\omega}^1 \cdot \mathbf{e}_\theta = 0, \quad \boldsymbol{\omega}^1 \cdot \mathbf{e}_r = 0, \\ \mathbf{v}^1 \cdot \mathbf{e}_R - \bar{\mathbf{v}}^1 \cdot \mathbf{e}_R = 0, \quad \mathbf{v}^1 \cdot \mathbf{e}_\theta - \bar{\mathbf{v}}^1 \cdot \mathbf{e}_\theta + \boldsymbol{\omega}^1 \cdot R_0 \mathbf{e}_\phi = 0,\end{gathered} \tag{66}$$

where some of the constraints have been applied to simplify other constraint expressions, for example: $\boldsymbol{\omega}^1 \cdot \mathbf{g}^2 = \boldsymbol{\omega}^1 \cdot \mathbf{e}_\theta$. After the constraints are imposed, $\mathcal{M} = T^2$, the 2-torus (Figure 5(b)). Lagrange's prescription yields the constraint forces and moments as:

$$\begin{gathered}\mathbf{F}_c^1 = \mu_6 \mathbf{e}_R + \mu_7 \mathbf{e}_\theta, \qquad \mathbf{M}_c^1 = \mu_4 \mathbf{e}_\theta + \mu_5 \mathbf{e}_r + \mu_7 R_0 \mathbf{e}_\phi, \\ \bar{\mathbf{F}}_c^1 = \mu_1 \mathbf{E}_1 + \mu_2 \mathbf{E}_2 + \mu_3 \mathbf{E}_3 - \mu_6 \mathbf{e}_R - \mu_7 \mathbf{e}_\theta.\end{gathered} \tag{67}$$

The generalized forces are:

$$\begin{gathered}Q_1 = \mathbf{F}_a^1 \cdot R_0 \sin(\phi_e)\, \mathbf{e}_\theta + \mathbf{M}_a^1 \cdot \mathbf{E}_3, \qquad Q_2 = \mathbf{F}_a^1 \cdot R_0 \mathbf{e}_\phi, \\ Q_{(2+i)} = \mathbf{F}_a^1 \cdot \mathbf{E}_i + \bar{\mathbf{F}}_a^1 \cdot \mathbf{E}_i + \mu_i, \qquad (i = 1, 2, 3), \\ Q_6 = \mathbf{M}_a^1 \cdot \mathbf{e}_\theta + \mu_4, \qquad Q_7 = \mathbf{F}_a^1 \cdot (-R_0 \cos(\phi_e)\, \mathbf{e}_\theta) + \mathbf{M}_a^1 \cdot \mathbf{e}_r + \mu_5, \\ Q_8 = \mathbf{F}_a^1 \cdot \mathbf{e}_R + \mu_6, \qquad Q_9 = \mathbf{F}_a^1 \cdot R_0 \sin(\phi_e)\, \mathbf{e}_\theta + \mu_7 R_0 \sin(\phi_e).\end{gathered} \tag{68}$$

The constraint forces and moments ensure that the representative particle remains on T^2. To obtain the equations of motion, we may impose the constraints on the velocities:

$$\mathbf{v}^1 = \dot{\nu}^1 R_0 \sin(\phi_e)\, \mathbf{e}_\theta + \dot{\phi}_e R_0 \mathbf{e}_\phi, \quad \bar{\mathbf{v}}^1 = \mathbf{0}, \quad \boldsymbol{\omega}^1 = \dot{\nu}^1 \mathbf{E}_3. \tag{69}$$

We have a covariant basis for $T_P T^2$ as

$$\begin{gathered}\mathbf{a}_1 = \frac{\partial \mathbf{r}}{\partial \nu^1} = \left(R_0 \sin(\phi_e)\, \mathbf{e}_\theta, \mathbf{0}, \operatorname{skwt}(\mathbf{E}_3)\, \mathbf{Q}^1 \right), \\ \mathbf{a}_2 = \frac{\partial \mathbf{r}}{\partial \phi_e} = \left(R_0 \mathbf{e}_\phi, \mathbf{0}, \mathbf{O} \right).\end{gathered} \tag{70}$$

The covariant basis spans a 2-dimensional vector space for all values of $\mathsf{q} = \left[\nu^1, \phi_e\right]^T$. Therefore, the coordinate system is singularity-free. If we let the gimbal have a radius of gyration k about $\mathbf{E}_3$ through O, we have the constrained kinetic energy of the physical system and the representative particle as:

$$T = \frac{1}{2} m^1 \left(\left(\dot{\nu}^1\right)^2 R_0^2 \sin^2(\phi_e) + \dot{\phi}_e^2 R_0^2 \right) + \frac{1}{2} M^1 k^2 \left(\dot{\nu}^1\right)^2, \tag{71}$$

which is a non-degenerate quadratic form for all q. The equations of motion are

$$\begin{bmatrix} m^1 R_0^2 \sin^2(\phi_e) + M^1 k^2 & 0 \\ 0 & m^1 R_0^2 \end{bmatrix} \begin{bmatrix} \ddot{\nu}^1 \\ \ddot{\phi}_e \end{bmatrix} + m^1 \begin{bmatrix} 2\dot{\nu}^1 \dot{\phi}_e \left(R_0^2 \sin(\phi_e) \cos(\phi_e) \right) \\ \left(\dot{\nu}^1\right)^2 \left(-R_0^2 \sin(\phi_e) \cos(\phi_e) \right) \end{bmatrix} = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}. \tag{72}$$

The presence of the moment of inertia $M^1 k^2$ makes the mass matrix (and the kinetic energy) non-singular. The quantity $M^1 k^2$ may be thought to control the outer radius of the configuration manifold, T^2. When $M^1k^2 \to 0$, T^2 degenerates into S^2, the 2-sphere, and $\{\nu^1, \phi_e\}$ become the usual extended spherical polar coordinates on S^2, complete with singularities at $\phi_e = 0, \pi$.

At $\phi_e = 0, \pi$, the particle comes into line with the axis of rotation of the gimbal. When at this location, a force applied to the particle in $\mathbf{e}_\theta$ has no affect on the motion of the system: it does not contribute to Q_1 or Q_2. Instead, it gets absorbed into $Q_3, \ldots, Q_9$ and is resisted entirely by constraint forces and moments. There are two circles in T^2 where the system experiences gimbal lock. At a point anywhere on one of these circles, if $\mathbf{F}_a^1 = F_a^1 \mathbf{e}_\theta$, then the corresponding $\mathbf{\Phi}_a$ is orthogonal to T^2.

To describe a particle on a half-gimbal, as shown in Figure 5(b), we use $\phi \in [0, \pi]$. Since O is a fixed point, the equations of motion are identical to (72), but with $\phi_e \mapsto \phi$ and all forces and moments are relative to point O. Two unilateral constraints $\phi \geq 0$ and $\phi \leq 0$ are imposed to produce a configuration manifold with two boundaries. In this case, it is the cylinder, $I \times S^1$ (Figure 5(d)). We require two constraint impulses applied to the particle at $\phi = 0$ and $\phi = \pi$ to enforce the two unilateral constraints. An extension of ϕ into ϕ_e, a cyclic coordinate, corresponds to gluing the boundaries of $I \times S^1$ together, resulting in T^2, the configuration manifold for a full gimbal.

The effect of unilateral constraints on the phase portrait of a particle in a gimbal controlled to rotate at a constant rate $\dot{\nu}^1 = \varOmega$ is shown in Figure 6. The equations of motion for the particle on the full gimbal and the half-gimbal under the influence of gravitational acceleration g are

$$\ddot{\phi}_e = \left(\varOmega^2 \cos(\phi_e) + \frac{g}{R_0}\right) \sin(\phi_e),$$
$$\ddot{\phi} = \left(\varOmega^2 \cos(\phi) + \frac{g}{R_0}\right) \sin(\phi), \tag{73}$$

respectively. For the half-gimbal, we assume an instantaneous perfectly elastic impact at $\phi = 0, \pi$ and that the impulsive force resulting from the impact is far more significant than any other forces applied to the particle. Then,

$$\dot{\phi}^+ = -\dot{\phi}^- \quad \text{at } \phi = 0 \text{ and } \phi = \pi, \tag{74}$$

where $\dot{\phi}^-$ and $\dot{\phi}^+$ are the values of $\dot{\phi}$ immediately before and after the collision, respectively.

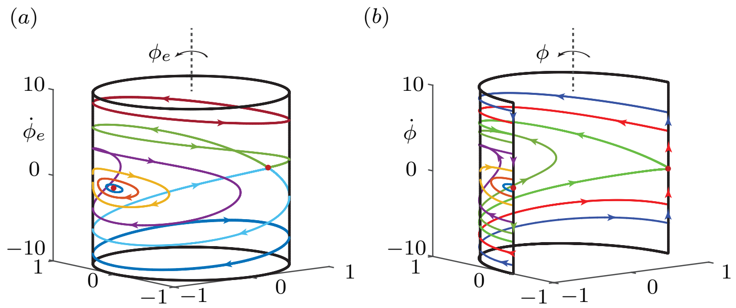

Fig. 6: *Phase portrait of solutions to (a) equation (73)$_1$ plotted on $I \times S^1$ and (b) equation (73)$_2$ plotted on $I \times I$.*

6 Rigid Body in a Cardan Suspension

In this section, we explore the Euler angle singularity and see how it is avoided through the imposition of constraints and the addition of system inertias. It has been shown that no global three-parameter representation of the rotation group exists without singularities [18], and the Euler angles are no exception. We will also show that gimbal lock once again corresponds to an applied loading that is orthogonal to the configuration manifold.

6.1 A Body Free to Rotate About a Fixed Point

Suppose a body is constrained so that its center of mass is a fixed point. Then, $\bar{\mathbf{x}}^1 = \mathbf{0}$ and the body engages in fixed-point rotation about its center of mass. The position vector to the representative particle in $\mathcal{C}^{12}$ is then

$$\mathbf{r} = \left(\mathbf{0}, \mathbf{Q}^1\right), \tag{75}$$

where $\mathbf{Q}^1$ is the rotation tensor of the body. The position vector $\mathbf{r}$ locates a point in $\mathcal{M} = \mathbb{R}P^3$ and the Euler angles $\left\{\nu^1, \nu^2, \nu^3\right\}$ provide a parametrization of this space. The tangent space to $\mathbb{R}P^3$ is Skw. A covariant basis for the tangent space is obtained as

$$\mathbf{a}_1 = \frac{\partial \mathbf{r}}{\partial \nu^1} = \left(\mathbf{0}, \text{skwt}\left(\mathbf{g}_1\right)\mathbf{Q}^1\right), \qquad \mathbf{a}_2 = \frac{\partial \mathbf{r}}{\partial \nu^2} = \left(\mathbf{0}, \text{skwt}\left(\mathbf{g}_2\right)\mathbf{Q}^1\right),$$
$$\mathbf{a}_3 = \frac{\partial \mathbf{r}}{\partial \nu^3} = \left(\mathbf{0}, \text{skwt}\left(\mathbf{g}_3\right)\mathbf{Q}^1\right). \tag{76}$$

For every Euler angle parametrization there are two values of ν^2 for which $\mathbf{g}_1 = \mathbf{g}_3$ and $\mathbf{g}_1 = -\mathbf{g}_3$. For example, the Euler basis for the 3-1-3 set is expressed in linear combinations of the inertial basis as

$$\begin{bmatrix}\mathbf{g}_1\\ \mathbf{g}_2\\ \mathbf{g}_3\end{bmatrix} = \begin{bmatrix} 0 & 0 & 1\\ \cos\left(\nu^1\right) & \sin\left(\nu^1\right) & 0\\ \sin\left(\nu^2\right)\sin\left(\nu^1\right) & -\sin\left(\nu^2\right)\cos\left(\nu^1\right) & \cos\left(\nu^2\right)\end{bmatrix}\begin{bmatrix}\mathbf{E}_1\\ \mathbf{E}_2\\ \mathbf{E}_3\end{bmatrix}. \tag{77}$$

At $\nu^2 = 0$, $\mathbf{g}_1 = \mathbf{g}_3$ and at $\nu^2 = \pi$, $\mathbf{g}_1 = -\mathbf{g}_3$. One can also show by substituting these critical values into expression (111) that there are ambiquities in the value of $\nu^1 + \nu^3$ or $\nu^1 - \nu^3$. An important consequence that occurs when $\mathbf{g}_1 = \pm\mathbf{g}_3$ is the failure of the set $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ span Skw, indicating that the Euler angle parametrization is corrupted by singularities. Using the isomorphism between Skw and $\mathbb{E}^3$, it is equivalent to say that the Euler basis $\{\mathbf{g}_i\}$ fails to capture every $\boldsymbol{\omega}^1 \in \mathbb{E}^3$ at the Euler angle singularity.

To illustrate the effect of the singularity on the parametrization of the kinetic energy and the equations of motion, consider a homogeneous spherically symmetric body with moment of inertia tensor $\mathbf{J}^1 = \lambda^1\mathbf{I}$ about its center of mass. The constrained kinetic energy of the body is

$$T = \frac{\lambda^1}{2}\left(\left(\dot{\nu}^1\right)^2 + \left(\dot{\nu}^2\right)^2 + \left(\dot{\nu}^3\right)^2 + 2\dot{\nu}^1\dot{\nu}^3\cos\left(\nu^2\right)\right), \tag{78}$$

which fails to be a positive-definite quadratic form in $\dot{\mathsf{q}} = \left[\dot{\nu}^1, \dot{\nu}^2, \dot{\nu}^3\right]^T$ at $\nu^2 = 0, \pi$. The equations of motion are

$$\lambda^1 \begin{bmatrix} 1 & 0 & \cos\left(\nu^2\right) \\ 0 & 1 & 0 \\ \cos\left(\nu^2\right) & 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{\nu}^1 \\ \ddot{\nu}^2 \\ \ddot{\nu}^3 \end{bmatrix} - \lambda^1 \sin\left(\nu^2\right) \begin{bmatrix} \dot{\nu}^2\dot{\nu}^3 \\ \dot{\nu}^3\dot{\nu}^1 \\ \dot{\nu}^1\dot{\nu}^2 \end{bmatrix} = \begin{bmatrix} \mathbf{M}^1 \cdot \mathbf{g}_1 \\ \mathbf{M}^1 \cdot \mathbf{g}_2 \\ \mathbf{M}^1 \cdot \mathbf{g}_3 \end{bmatrix}. \tag{79}$$

The mass matrix M is non-invertible at $\nu^2 = 0, \pi$. Notice also at the singularity that a moment applied in the $\mathbf{g}_1 \times \mathbf{g}_2$-direction is not captured by the right hand side of equation (79). This result is due to the coordinate singularity and not due to gimbal lock. For the representative particle, we have the equivalence $Q_i = \mathbf{\Phi} \cdot \mathbf{a}_i = \mathbf{M}^1 \cdot \mathbf{g}_i$, where $\mathbf{\Phi}$ is a vector in $\mathcal{C}^{12}$ involving the skew-symmetric tensor of the applied moment $\mathbf{M}^1$. For further details, see [6].

6.2 Imposition of Constraints

Suppose ν^1 is to be constrained such that

$$\nu^1 - f(t) = 0. \tag{80}$$

For the 3-1-3 Euler angle set, the dual Euler basis has the representation

$$\begin{bmatrix} \mathbf{g}^1 \\ \mathbf{g}^2 \\ \mathbf{g}^3 \end{bmatrix} = \begin{bmatrix} -\sin\left(\nu^1\right)\cot\left(\nu_e^2\right) & \cos\left(\nu^1\right)\cot\left(\nu_e^2\right) & 1 \\ \cos\left(\nu^1\right) & \sin\left(\nu^1\right) & 0 \\ \sin\left(\nu^1\right)\csc\left(\nu_e^2\right) & -\cos\left(\nu^1\right)\csc\left(\nu_e^2\right) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}, \tag{81}$$

where $\nu^2 \in [0, \pi]$ has been extended to $\nu_e^2 \in [0, 2\pi)$ (cf. Section A.1). We may write constraint (80) in terms of the angular velocity vector as

$$\boldsymbol{\omega}^1 \cdot \mathbf{g}^1 - \dot{f}(t) = 0. \tag{82}$$

Lagrange's prescription yields an expression for the required constraint moment as

$$\mathbf{M}_c^1 = \mu \mathbf{g}^1. \tag{83}$$

The constrained angular velocity vector is

$$\boldsymbol{\omega}^1 = \dot{\nu}_e^2 \mathbf{g}_2 + \dot{\nu}^3 \mathbf{g}_3 + \dot{f} \mathbf{g}_1. \tag{84}$$

The generalized coordinates in this case are $\left\{\nu_e^2, \nu^3\right\}$ and the configuration manifold is the 2-torus, T^2. A covariant basis for the tangent space $T_P T^2$ is

$$\mathbf{a}_1 = \frac{\partial \mathbf{r}}{\partial \nu_e^2} = \left(\mathbf{0}, \operatorname{skwt}\left(\mathbf{g}_2\right)\mathbf{Q}^1\right), \qquad \mathbf{a}_2 = \frac{\partial \mathbf{r}}{\partial \nu^3} = \left(\mathbf{0}, \operatorname{skwt}\left(\mathbf{g}_3\right)\mathbf{Q}^1\right). \tag{85}$$

Since $\mathbf{g}_2 \cdot \mathbf{g}_3 = 0$ always, the coordinate system is singularity-free. The constrained kinetic energy is given by

$$T = T_2 + T_1 + T_0 = \frac{1}{2}\lambda^1 \left(\left(\dot{\nu}_e^2\right)^2 + \left(\dot{\nu}^3\right)^2\right) + \lambda^1 \dot{f} \dot{\nu}^3 \cos\left(\nu_e^2\right) + \frac{1}{2}\lambda^1 \dot{f}^2. \tag{86}$$

Here, T_2 is a well-defined positive-definite quadratic form in $\dot{\mathsf{q}} = \left[\dot{\nu}_e^2, \dot{\nu}^3\right]^T$ for all q. The equations of motion resulting from equation (79) after the constraint is imposed are

$$\lambda^1 \begin{bmatrix} \ddot{\nu}_e^2 \\ \ddot{\nu}^3 \end{bmatrix} - \lambda^1 \sin\left(\nu_e^2\right) \begin{bmatrix} \dot{f}\dot{\nu}^3 \\ \dot{f}\dot{\nu}_e^2 \end{bmatrix} = \begin{bmatrix} \mathbf{M}_a^1 \cdot \mathbf{g}_2 \\ \mathbf{M}_a^1 \cdot \mathbf{g}_3 - \lambda^1 \ddot{f} \cos\left(\nu_e^2\right) \end{bmatrix}. \tag{87}$$

with

$$\mathbf{M}_c^1 = \left(\lambda^1 \ddot{f} + \lambda^1 \cos\left(\nu_e^2\right) \ddot{\nu}^3 - \lambda^1 \sin\left(\nu_e^2\right) \dot{\nu}_e^2 \dot{\nu}^3 - \mathbf{M}_a^1 \cdot \mathbf{g}_1\right) \mathbf{g}^1, \tag{88}$$

where $\mathbf{M}_a^1$ is the applied non-constraint moment. The mass matrix is invertible and a non-singular expression for the generalized coordinate accelerations may be obtained. Utilizing the second equation of motion for $\ddot{\nu}^3$ and expression (81) for the dual Euler basis, one can show

$$\mathbf{M}_c^1 = \lambda^1 \left(\dot{\nu}_e^2 \dot{f} \cos\left(\nu_e^2\right) - \dot{\nu}_e^2 \dot{\nu}^3 + \ddot{f} \sin\left(\nu_e^2\right)\right) \mathbf{u} - \left(\mathbf{M}_a^1 \cdot \mathbf{u}\right) \mathbf{u}, \tag{89}$$

where

$$\mathbf{u} = \cos\left(\nu_e^2\right) \left(-\sin(f)\,\mathbf{E}_1 + \cos(f)\,\mathbf{E}_2\right) + \sin\left(\nu_e^2\right) \mathbf{E}_3. \tag{90}$$

The constraint moment, $\mathbf{M}_c^1 = \mu \mathbf{g}^1$ is therefore well-defined for all ν_e^2, despite $\mathbf{g}^1$ becoming infinite at $\nu_e^2 = 0, \pi$. The basis $\{\mathbf{u}, \mathbf{g}_2, \mathbf{g}_3\}$ is orthonormal and is suitable for capturing any applied moment that may contribute to the motion or the constraint moment. If we chose to inherit $\nu^2 \in [0, \pi]$ from the parametrization of the ambient space, $\mathbb{R}P^3$, then two unilateral constraints are required: $\nu^2 \geq 0$ and $\nu^2 \leq \pi$. At these two values of ν^2, there is a constraint angular impulse of the form

$$\hat{\mathbf{M}}_c^1 = [[\mathbf{H}^1]] = \mathbf{H}^{1+} - \mathbf{H}^{1-} = \lambda^1 \left(\boldsymbol{\omega}^{1+} - \boldsymbol{\omega}^{1-}\right), \tag{91}$$

where $\mathbf{H}^{1-}$ and $\mathbf{H}^{1+}$ are the angular momenta immediately before and after collision, respectively. The configuration manifold becomes $I \times S^1$, a manifold with two boundaries owing to the imposition of the two unilateral constraints.

6.3 Adding System Inertias

Consider a system of three rigid bodies, $\mathcal{B}^1, \mathcal{B}^2$, and $\mathcal{B}^3$ constrained in such a way as to have the reference configuration in Figure 7(a). Without a loss in generality, we assume all bodies to exhibit spherical symmetry so that the moment of inertia tensor about the center of mass of the Ith body is $\mathbf{J}^I = \lambda^I \mathbf{I}$. In the engineering design of inertial measurement units, $\mathcal{B}^3$ assumes the role of a stable platform while $\mathcal{B}^1$ and $\mathcal{B}^2$ are spherical gimbal mounts, and $\mathcal{B}^1$ is attached to a freely rotating spacecraft. This particular arrangement of gimbals is known as the Cardan suspension. In the subsequent derivation, we assume the space outside of the gimbals to be inertially fixed. Our analysis is similar to that found in [1, Section 11.8] for a rotating platform with vehicle fixed.

The position vector of the representative particle for the unconstrained system in $\mathcal{C}^{36}$ is

$$\mathbf{r} = \left(\bar{\mathbf{x}}^1, \bar{\mathbf{x}}^2, \bar{\mathbf{x}}^3, \mathbf{Q}^1, \mathbf{Q}^2, \mathbf{Q}^3\right). \tag{92}$$

Let $\left\{\alpha^1, \alpha^2, \alpha^3\right\}$ be a set of 3-2-1 Euler angles for $\mathbf{Q}^1$, $\left\{\beta^1, \beta^2, \beta^3\right\}$ a set of 1-3-2 Euler angles for the relative rotation tensor $\mathbf{Q}^2\left(\mathbf{Q}^1\right)^T$, and $\left\{\gamma^1, \gamma^2, \gamma^3\right\}$ a set of 3-2-1

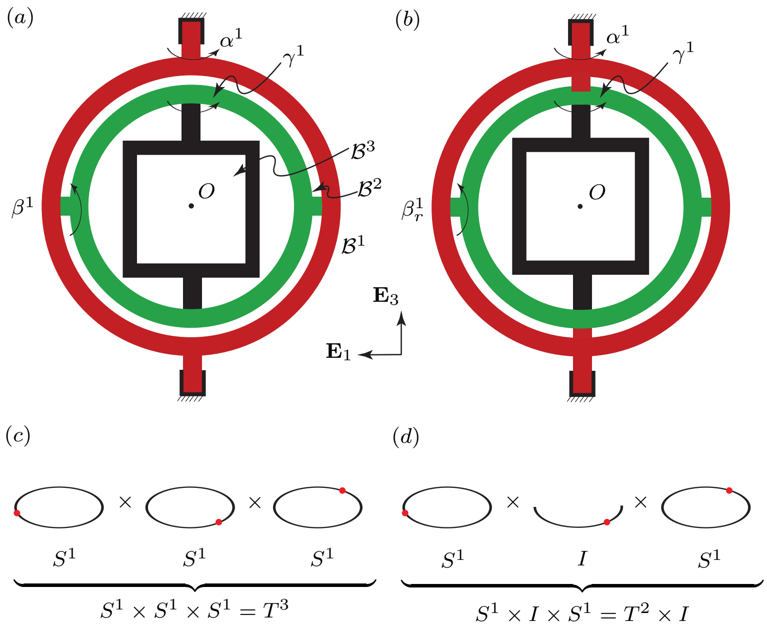

Fig. 7: *A system of (a) three nested gimbals has (c) the 3-torus, T^3 as its configuration manifold. When two unilateral constraints on the middle angle are imposed as in (b), the configuration manifold develops two boundaries and becomes (d) $T^2 \times I$.*

Euler angles for the relative rotation tensor $\mathbf{Q}^3\left(\mathbf{Q}^2\right)^T$. If we use rectangular Cartesian coordinates for $\bar{\mathbf{x}}^I$, then the parametrizations of $\mathbf{Q}^I$ are independent of those for $\bar{\mathbf{x}}^I$, and we may focus our attention on the 27-dimensional subspace of $\mathcal{C}^{36}$ containing just the 3-tuples $\left(\mathbf{Q}^1, \mathbf{Q}^2, \mathbf{Q}^3\right)$. Denoting the Euler basis for each body as $\left\{\mathbf{g}_i^I\right\}$, the unconstrained angular velocity vectors are expressible as

$$\boldsymbol{\omega}^1 = \sum_{i=1}^{3} \dot{\alpha}^i \mathbf{g}_i^1, \quad \boldsymbol{\omega}^2 - \boldsymbol{\omega}^1 = \sum_{i=1}^{3} \dot{\beta}^i \mathbf{g}_i^2, \quad \boldsymbol{\omega}^3 - \boldsymbol{\omega}^2 = \sum_{i=1}^{3} \dot{\gamma}^i \mathbf{g}_i^3. \tag{93}$$

The integrable constraints on the system are

$$\alpha^2 = 0, \quad \alpha^3 = 0, \quad \beta^2 = 0, \quad \beta^3 = 0, \quad \gamma^2 = 0, \quad \gamma^3 = 0. \tag{94}$$

A corotational basis for the Ith body is denoted $\left\{\mathbf{e}_i^I\right\}$, so that $\mathbf{e}_i^I = \mathbf{Q}^I \mathbf{E}_i$, while the dual Euler basis is denoted $\left\{\mathbf{g}^{iI}\right\}$. The constraints imply that the Euler basis and dual Euler basis for each body are orthonormal (though perhaps left-handed):

$$\begin{aligned} \mathbf{g}_2^1 &= \mathbf{g}^{21} = \mathbf{e}_2^1, \quad \mathbf{g}_3^1 = \mathbf{g}^{31} = \mathbf{e}_1^1, \quad \mathbf{g}_2^2 = \mathbf{g}^{22} = \mathbf{e}_3^2, \\ \mathbf{g}_3^2 &= \mathbf{g}^{32} = \mathbf{e}_2^2, \quad \mathbf{g}_2^3 = \mathbf{g}^{23} = \mathbf{e}_2^3, \quad \mathbf{g}_3^3 = \mathbf{g}^{33} = \mathbf{e}_1^3, \end{aligned} \tag{95}$$

The axes of suspension are

$$\mathbf{g}_1^1 = \mathbf{g}^{11} = \mathbf{E}_3, \quad \mathbf{g}_1^2 = \mathbf{g}^{12} = \mathbf{e}_1^2, \quad \mathbf{g}_1^3 = \mathbf{g}^{13} = \mathbf{e}_3^3. \tag{96}$$

The constraints may be expressed in terms of the angular velocity vectors as

$$\boldsymbol{\omega}^1 \cdot \mathbf{e}_2^1 = 0, \quad \boldsymbol{\omega}^1 \cdot \mathbf{e}_1^1, \quad \left(\boldsymbol{\omega}^2 - \boldsymbol{\omega}^1\right) \cdot \mathbf{e}_3^2 = 0,$$
$$\left(\boldsymbol{\omega}^2 - \boldsymbol{\omega}^1\right) \cdot \mathbf{e}_2^2 = 0, \quad \left(\boldsymbol{\omega}^3 - \boldsymbol{\omega}^2\right) \cdot \mathbf{e}_2^3 = 0, \quad \left(\boldsymbol{\omega}^3 - \boldsymbol{\omega}^2\right) \cdot \mathbf{e}_1^3 = 0. \tag{97}$$

Lagrange's prescription gives the constraint moments acting on each body about O as

$$\mathbf{M}_c^1 = \mu_1 \mathbf{e}_2^1 + \mu_2 \mathbf{e}_1^1 - \mu_3 \mathbf{e}_3^2 - \mu_4 \mathbf{e}_2^2, \qquad \mathbf{M}_c^2 = \mu_3 \mathbf{e}_3^2 + \mu_4 \mathbf{e}_2^2 - \mu_5 \mathbf{e}_2^3 - \mu_6 \mathbf{e}_1^3,$$
$$\mathbf{M}_c^3 = \mu_5 \mathbf{e}_2^3 + \mu_6 \mathbf{e}_1^3. \tag{98}$$

The generalized coordinates are $\left\{q^1, q^2, q^3\right\} = \left\{\alpha^1, \beta^1, \gamma^1\right\}$, while the constrained coordinates are $\left\{q^4, \ldots, q^9\right\} = \left\{\alpha^2, \alpha^3, \beta^2, \beta^3, \gamma^2, \gamma^3\right\}$. We note the following sets:

$$\left\{\frac{\partial \boldsymbol{\omega}^1}{\partial \dot{q}^J}\right\} = \left\{\mathbf{E}_3, \mathbf{0}, \mathbf{0}, \mathbf{e}_2^1, \mathbf{e}_1^1, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}\right\}, \qquad \left\{\frac{\partial \boldsymbol{\omega}^2}{\partial \dot{q}^J}\right\} = \left\{\mathbf{E}_3, \mathbf{e}_1^2, \mathbf{0}, \mathbf{e}_2^1, \mathbf{e}_1^1, \mathbf{e}_3^2, \mathbf{e}_2^2, \mathbf{0}, \mathbf{0}\right\},$$
$$\left\{\frac{\partial \boldsymbol{\omega}^3}{\partial \dot{q}^J}\right\} = \left\{\mathbf{E}_3, \mathbf{e}_1^2, \mathbf{e}_3^3, \mathbf{e}_2^1, \mathbf{e}_1^1, \mathbf{e}_3^2, \mathbf{e}_2^2, \mathbf{e}_2^3, \mathbf{e}_1^3\right\}. \tag{99}$$

The configuration manifold in this case is the 3-torus, T^3 as in Figure 7(c). A basis for the tangent space $T_P T^3$ is gotten as

$$\mathbf{a}_1 = \frac{\partial \mathbf{r}}{\partial \alpha^1} = \left(\operatorname{skwt}\left(\mathbf{E}_3\right) \mathbf{Q}^1, \frac{\partial \mathbf{Q}^2}{\partial \alpha^1}, \frac{\partial \mathbf{Q}^3}{\partial \alpha^1}\right), \qquad \mathbf{a}_2 = \frac{\partial \mathbf{r}}{\partial \beta^1} = \left(\mathbf{O}, \frac{\partial \mathbf{Q}^2}{\partial \beta^1}, \frac{\partial \mathbf{Q}^3}{\partial \beta^1}\right),$$
$$\mathbf{a}_3 = \frac{\partial \mathbf{r}}{\partial \gamma^1} = \left(\mathbf{O}, \mathbf{O}, \frac{\partial \mathbf{Q}^3}{\partial \gamma^1}\right). \tag{100}$$

The set $\{\mathbf{a}_i\}$ is linearly independent for all values of $\mathsf{q} = \left[\alpha^1, \beta^1, \gamma^1\right]^T$, and the coordinate system is therefore singularity-free on T^3. The generalized forces are

$$Q_1 = \left(\mathbf{M}_a^1 + \mathbf{M}_a^2 + \mathbf{M}_a^3\right) \cdot \mathbf{E}_3, \qquad Q_2 = \left(\mathbf{M}_a^2 + \mathbf{M}_a^3\right) \cdot \mathbf{e}_1^2, \qquad Q_3 = \mathbf{M}_a^3 \cdot \mathbf{e}_3^3,$$
$$Q_4 = \left(\mathbf{M}_a^1 + \mathbf{M}_a^2 + \mathbf{M}_a^3\right) \cdot \mathbf{e}_2^1 + \mu_1, \qquad Q_5 = \left(\mathbf{M}_a^1 + \mathbf{M}_a^2 + \mathbf{M}_a^3\right) \cdot \mathbf{e}_1^1 + \mu_2,$$
$$Q_6 = \left(\mathbf{M}_a^2 + \mathbf{M}_a^3\right) \cdot \mathbf{e}_3^2 + \mu_3, \qquad Q_7 = \left(\mathbf{M}_a^2 + \mathbf{M}_a^3\right) \cdot \mathbf{e}_2^2 + \mu_4,$$
$$Q_8 = \mathbf{M}_a^3 \cdot \mathbf{e}_2^3 + \mu_5, \qquad Q_9 = \mathbf{M}_a^3 \cdot \mathbf{e}_1^3 + \mu_6. \tag{101}$$

The constrained kinetic energy of the system is

$$T = T_2 = \frac{1}{2}\lambda^1 \left(\dot{\alpha}^1\right)^2 + \frac{1}{2}\lambda^2 \left(\left(\dot{\alpha}^1\right)^2 + \left(\dot{\beta}^1\right)^2\right)$$
$$+ \frac{1}{2}\lambda^3 \left(\left(\dot{\alpha}^1\right)^2 + \left(\dot{\beta}^1\right)^2 + \left(\dot{\gamma}^1\right)^2 + 2\dot{\alpha}^1 \dot{\gamma}^1 \cos\left(\beta^1\right)\right), \tag{102}$$

which is a non-degenerate positive-semidefinite quadratic form in $\dot{\mathsf{q}}$ for all q. The resulting equations of motion are

$$\begin{bmatrix} \lambda^1 + \lambda^2 + \lambda^3 & 0 & \lambda^3 \cos\left(\beta^1\right) \\ 0 & \lambda^2 + \lambda^3 & 0 \\ \lambda^3 \cos\left(\beta^1\right) & 0 & \lambda^3 \end{bmatrix} \begin{bmatrix} \ddot{\alpha}^1 \\ \ddot{\beta}^1 \\ \ddot{\gamma}^1 \end{bmatrix} + \lambda^3 \sin\left(\beta^1\right) \begin{bmatrix} -\dot{\beta}^1 \dot{\gamma}^1 \\ \dot{\alpha}^1 \dot{\gamma}^1 \\ -\dot{\alpha}^1 \dot{\beta}^1 \end{bmatrix} = \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix}. \tag{103}$$

For nonzero gimbal inertias λ^1 and/or λ^2, the mass matrix is invertible. When $\beta^1 = 0, \pi$, a moment applied to $\mathcal{B}^3$ in the $\mathbf{e}_2^2$-direction contributes nothing to Q_1, Q_2, or Q_3. Instead, it gets absorbed by $Q_4, \ldots, Q_9$ and contributes to the constraint moments. Gimbal lock occurs everywhere on the configuration manifold where $\beta^1 = 0$ and $\beta^1 = \pi$. At these values, the applied force $\mathbf{\Phi}_a$ on the representative particle corresponding to an applied moment $\mathbf{M}_a^3 = \left(\mathbf{M}_a^3 \cdot \mathbf{e}_2^2\right) \mathbf{e}_2^2$ is perfectly orthogonal to T^3.

When we allow $\lambda^1 = 0$ and $\lambda^2 = 0$, the configuration manifold degenerates from T^3 into $\mathbb{R}P^3$, and the set $\left\{\alpha^1, \beta^1, \gamma^1\right\}$ become a set of ordinary extended 3-1-3 Euler angles for the rotation tensor of $\mathcal{B}^3$, and the singularities reappear. If one wishes to insert mechanical stops into the system as in Figure 7(b), then the equations of motion and constraint force prescriptions remain the same but a restricted value of $\beta_r^1 \in [0, \pi]$ is required. The unilateral constraints $\beta_r^1 \geq 0$ and $\beta_r^1 \leq 0$ would be enforced by constraint impulse moments, as discussed previously. The configuration manifold develops two boundaries and becomes $T^2 \times I$, as shown in Figure 7(d).

7 Conclusions

In this paper, we have illustrated using various examples from the dynamics of particles, the notions of coordinate singularities and gimbal lock. In particular, a coordinate singularity occurs when the covariant basis induced by the chosen coordinates fails to span the tangent space of the configuration manifold. We have also demonstrated that gimbal lock occurs when a system of applied forces and moments is orthogonal to the configuration manifold of a system. It can also be shown that the locking of the Cardan joint mentioned in the introduction is an example of the orthogonality of the applied forces and moments.

To remove a coordinate singularity, constraints can be imposed or additional components can be added to a mechanical system. As illustrated in the examples shown in this paper, to achieve these effects, the additions serve to change the topology of the configuration manifold. In cases involving unilateral constraints, such as the particle in a gimbal shown in Figure 5(b), the configuration manifold can develop a boundary and constraint impulses are required to keep the representative particle on the manifold.

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A Appendix Background on Rotations

An example of a 3-dimensional configuration manifold results from the constrained motion of a rigid body rotating about a fixed point, such as in the motion of the Lagrange top. In this case, the rotation tensor $\mathbf{Q}$ of the top belongs to the proper-orthogonal subgroup of linear second-order tensors, a 3-dimensional manifold called Orth^+ embedded in $\mathcal{E}^9$. This subgroup is isomorphic to $SO(3)$, the special orthogonal group of 3×3 matrices. Furthermore, $SO(3)$ is diffeomorphic to $\mathbb{R}P^3$, real projective 3-space, and we can associate the configuration of the top to a point in $\mathbb{R}P^3$.[4]

Given an angle θ and axis of rotation $\mathbf{p}$, Euler's formula for the rotation tensor is

$$\mathbf{Q} = \mathbf{L}\left(\theta, \mathbf{p}\right) = \cos\left(\theta\right)\left(\mathbf{I} - \mathbf{p}\otimes\mathbf{p}\right) + \sin\left(\theta\right)\text{skwt}\left(\mathbf{p}\right) + \mathbf{p}\otimes\mathbf{p}, \tag{104}$$

where skwt $(\mathbf{p})$ indicates the skew tensor of $\mathbf{p}$, which is given by the mapping

$$\text{skwt}\left(\mathbf{p}\right) = -\boldsymbol{\epsilon}\mathbf{p}, \tag{105}$$

[4] For representations of rotations with constant angular velocities on the real projective 2-space $\mathbb{R}P^2$, the reader is referred to [12].

where $\boldsymbol{\epsilon}$ is the third-order alternator tensor. Since $\mathbf{Q}$ is proper orthogonal, it satisfies $\mathbf{Q}\mathbf{Q}^T = \mathbf{I}$ and $\det \mathbf{Q} = +1$. A skew-symmetric angular velocity tensor may therefore be defined using $\mathbf{Q}$ as $\boldsymbol{\Omega} = \dot{\mathbf{Q}}\mathbf{Q}^T$. We denote the space of skew-symmetric tensors as Skw. The angular velocity vector $\boldsymbol{\omega}$ is the axial vector of $\boldsymbol{\Omega}$ which is given by the mapping

$$\boldsymbol{\omega} = \mathrm{ax}\left[\boldsymbol{\Omega}\right] = -\frac{1}{2}\boldsymbol{\epsilon}\left[\boldsymbol{\Omega}\right]. \tag{106}$$

Thus, there exists an isomorphism between Skw and $\mathbb{E}^3$ through the maps $\mathrm{ax}(\cdot)$ and $\mathrm{skwt}(\cdot)$ for which the following is true:

$$\boldsymbol{\Omega}\mathbf{a} = \boldsymbol{\omega} \times \mathbf{a} \tag{107}$$

for any $\mathbf{a} \in \mathbb{E}^3$. Euler angles are a common choice of parametrization for $\mathbf{Q}$. A general Euler angle decomposition is given by

$$\mathbf{Q} = \mathbf{L}\left(\nu^3, \mathbf{g}_3\right)\mathbf{L}\left(\nu^2, \mathbf{g}_2\right)\mathbf{L}\left(\nu^1, \mathbf{g}_1\right), \tag{108}$$

where $\left\{\nu^1, \nu^2, \nu^3\right\}$ are the Euler angles and $\{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\}$ are the corresponding axes of rotation, which are collectively known as the Euler basis [15, 14, 16]. It is also convenient to define a reciprocal basis $\left\{\mathbf{g}^i\right\}$ known as the dual Euler basis with the property

$$\mathbf{g}_i \cdot \mathbf{g}^j = \delta_i^j, \qquad (i, j = 1, 2, 3), \tag{109}$$

where δ_i^j is the Kronecker delta. A corotational (or body-fixed) basis is defined as the set $\{\mathbf{e}_i\}$ for which $\mathbf{e}_i = \mathbf{Q}\mathbf{E}_i$ for $i = 1, 2, 3$. The rotation tensor has the same components in the corotational basis as it does in the inertial basis: $\mathbf{Q} = Q_{ij}\mathbf{e}_i \otimes \mathbf{e}_j = Q_{ij}\mathbf{E}_i \otimes \mathbf{E}_j$. Using the abbreviations $\cos(\nu) = \mathrm{c}\nu$ and $\sin(\nu) = \mathrm{s}\nu$, a 3-2-1 Euler angle set yields the following parametrization:

$$[Q_{ij}] = \begin{bmatrix} \mathrm{c}\nu^1\mathrm{c}\nu^2 & \mathrm{c}\nu^1\mathrm{s}\nu^2\mathrm{s}\nu^3 - \mathrm{s}\nu^1\mathrm{c}\nu^3 & \mathrm{c}\nu^1\mathrm{s}\nu^2\mathrm{c}\nu^3 + \mathrm{s}\nu^1\mathrm{s}\nu^3 \\ \mathrm{s}\nu^1\mathrm{c}\nu^2 & \mathrm{s}\nu^1\mathrm{s}\nu^2\mathrm{s}\nu^3 + \mathrm{c}\nu^1\mathrm{c}\nu^3 & \mathrm{s}\nu^1\mathrm{s}\nu^2\mathrm{c}\nu^3 - \mathrm{c}\nu^1\mathrm{s}\nu^3 \\ -\mathrm{s}\nu^2 & \mathrm{c}\nu^2\mathrm{s}\nu^3 & \mathrm{c}\nu^2\mathrm{c}\nu^3 \end{bmatrix}, \tag{110}$$

where $\nu^1 \in [0, 2\pi)$, $\nu^2 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, and $\nu^3 \in [0, 2\pi)$. A 3-1-3 Euler angle set yields the following parametrization:

$$[Q_{ij}] = \begin{bmatrix} \mathrm{c}\nu^1\mathrm{c}\nu^3 - \mathrm{s}\nu^1\mathrm{c}\nu^2\mathrm{s}\nu^3 & -\mathrm{c}\nu^1\mathrm{s}\nu^3 - \mathrm{s}\nu^1\mathrm{c}\nu^2\mathrm{c}\nu^3 & \mathrm{s}\nu^1\mathrm{s}\nu^2 \\ \mathrm{s}\nu^1\mathrm{c}\nu^3 + \mathrm{c}\nu^1\mathrm{c}\nu^2\mathrm{s}\nu^3 & -\mathrm{s}\nu^1\mathrm{s}\nu^3 + \mathrm{c}\nu^1\mathrm{c}\nu^2\mathrm{c}\nu^3 & -\mathrm{c}\nu^1\mathrm{s}\nu^2 \\ \mathrm{s}\nu^2\mathrm{s}\nu^3 & \mathrm{s}\nu^2\mathrm{c}\nu^3 & \mathrm{c}\nu^2 \end{bmatrix}, \tag{111}$$

where $\nu^1 \in [0, 2\pi)$, $\nu^2 \in [0, \pi]$, and $\nu^3 \in [0, 2\pi)$.

For a given Euler angle parametrization, the following identity may be shown to hold:

$$\frac{\partial \mathbf{Q}}{\partial \nu^i}\mathbf{Q}^T = \mathrm{skwt}\left(\mathbf{g}_i\right), \quad (i = 1, 2, 3). \tag{112}$$

where $\mathbf{Q}$ is given by equation (108).

A.1 Constraining One of the Euler Angles

An interesting situation occurs when the first or third Euler angle is constrained and the configuration manifold for the rotational motion of the rigid body becomes a two-torus T^2 provided ν^2 is extended to $\nu_e^2 \in [0, 2\pi)$. The latter case arises in the motion of the rigid bodies shown in Figure 1.[5] If a set of 3-1-3 Euler angles are used to parameterize the rotation tensor $\mathbf{Q}$ of these bodies, then the third angle $\nu^3 = 0$. To use two angles to parameterize the rotation $\mathbf{Q}$ of the constrained rigid bodies shown in Figure 1, it is necessary that

$$\mathbf{Q} = \mathbf{L}\left(\nu_e^2, \mathbf{e}_1\right)\mathbf{L}\left(\nu^1, \mathbf{E}_3\right), \tag{113}$$

[5] A discussion of the constraint forces and moments acting on the bodies shown in this figure can be found in [16].

In contrast to the unconstrained case (108), it is necessary to extend the range of the second angle: $\nu^2 \to \nu_e^2 \in [0, 2\pi)$. The necessity of this extension can also be demonstrated by taking a pen and placing it on a horizontal surface. Fixing the orientation of the longitudinal axis of the pen is equivalent to fixing ν^1. Then, by rotating the pen about its longitudinal axis, the necessity of having the extended angle ν_e^2 can be readily seen.

A.2 A Fixed Axis of Rotation

Suppose $\mathbf{Q}$ is constrained so that the rotation axis is fixed: $\mathbf{p} = \mathbf{p}_0$. Then, the configuration manifold is the 1-dimensional circle S^1 contained in Orth^+: $\mathbf{p}_0 \otimes \mathbf{p}_0$ is normal to the plane of the circle and $\cos(\theta)(\mathbf{I} - \mathbf{p}_0 \otimes \mathbf{p}_0) + \sin(\theta)\,\mathrm{skwt}(\mathbf{p}_0)$ is the radius vector to a point on the circle. A tensor spanning the tangent space to S^1 is

$$\mathbf{a}_1 = \frac{\partial \mathbf{Q}}{\partial \theta} = -\sin(\theta)(\mathbf{I} - \mathbf{p}_0 \otimes \mathbf{p}_0) + \cos(\theta)\ \mathrm{skwt}(\mathbf{p}_0)\,, \tag{114}$$

which is clearly tangent to the circle.