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Perspectives on Euler Angle Singularities, Gimbal Lock, and the Orthogonality of Applied Forces and Applied Moments

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Abstract Coordinate singularities and gimbal lock are two phenomena that feature in models for the dynamics of mechanical systems. The former phenomenon pertains to the coordinates used to parameterize the configuration manifold of the system, while the latter phenomenon has a distinctive physical manifestation. In the present paper, we use tools from differential geometry to show how gimbal lock is intimately associated with an orthogonality condition on the applied forces and moments which act on the system. This condition is equivalent to a generalized applied force being normal to the configuration manifold of the system. Numerous examples, including the classic bead on a rotating hoop example and a gimbaled rigid body, are used to illuminate the orthogonality condition. These examples help to offer a new explanation for the elimination of gimbal lock by the addition of gimbals and demonstrate how integrable constraints alter the configuration manifold and may consequently eliminate coordinate singularities.

Keywords Gimbal Lock · Coordinate Singularities · Configuration Manifold · Differential Geometry · Analytical Mechanics

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1 Introduction

The phenomenon of gimbal lock rose in prominence during the time of NASA's Apollo program. Inside the inertial measurement unit of the Apollo spacecraft was a stable (inertial) platform about which the spacecraft rotated. The hinged platform was suspended in two gimbals to give it three degrees of freedom (cf. Figure 1(a)). Its role was to give an accurate measure of the spacecraft's orientation. At a particular orientation of the spacecraft, the gimbals became coplanar and control torques became ineffective at stabilizing the platform. Moreover, if the spacecraft then rotated about an axis normal to the common plane of the gimbals, the platform became "locked" and engaged in the same rotation as the spacecraft, thereby rendering it useless as an inertial reference. To correct for the locking of the gimbals, the spacecraft would have to pitch away from the problematic orientation and the platform would be reoriented relative to the stars. To avoid gimbal lock entirely, three gimbals could be used with a control scheme to keep three suspension axes 90° apart (cf. Figure 1(b)). For the Apollo program, the engineers, who were fully cognizant of gimbal lock, opted to use less gimbals to save on weight costs, and preferred instead that the astronauts navigate around gimbal lock.[1]

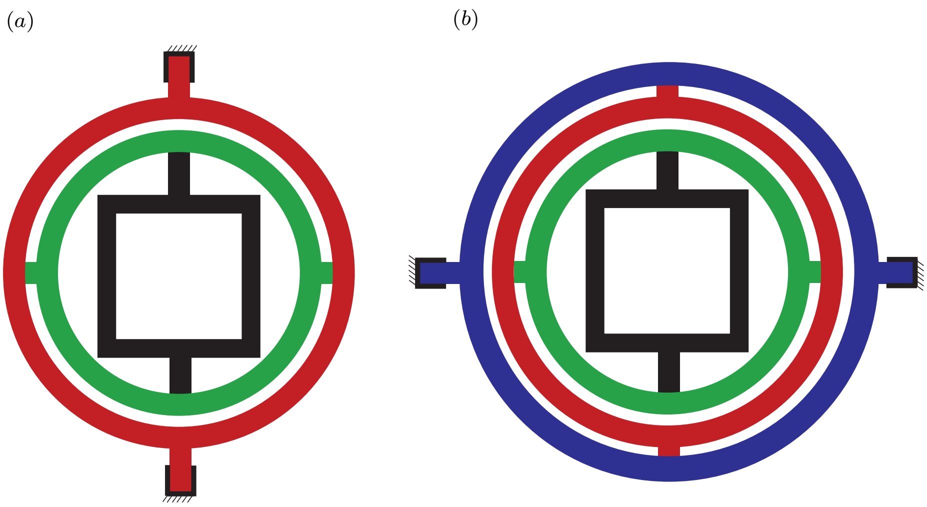

Fig. 1: *Example of a platform mounted in (a) two and (b) three gimbals for a three-axis and a four-axis suspension, respectively.*

Another phenomenon in the dynamics of rigid bodies, which is often (but not always) confounded with gimbal lock, is the singularity that occurs in the Euler angle parameterization of rotations. Related singularities arise when cylindrical polar coordinates and spherical polar coordinates are used to parameterize the motion of a particle.

[1] For a comprehensive reference on gyroscopic devices mechanically suspended in gimbals, see [1]. For a technical overview of the inertial platforms used in the Apollo spacecraft missions, see [19] and [29]. Other useful references on inertial navigation include [8], [21], and [30, Chapter 6].

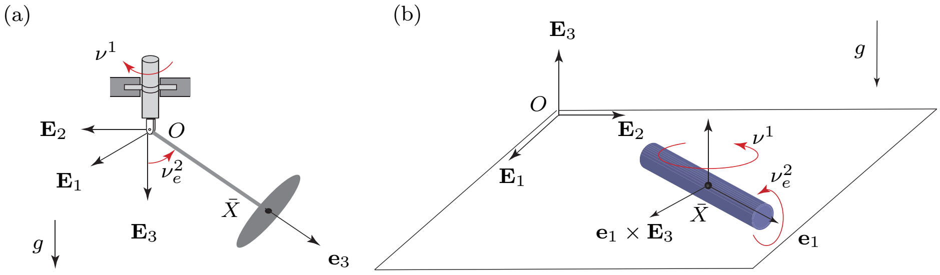

Fig. 2: *Examples of constrained rigid bodies described by non-degenerate coordinates. In (a), the configuration manifold of a pin-jointed whirling rigid body is a two-torus T^2 and in (b), the configuration manifold of a cylinder sliding on a stationary surface is a four-dimensional manifold, $\mathcal{E}^2 \times T^2$.*

What is not fully appreciated is that often these singularities are removed when the system is constrained. For instance, the motion of a rigid body freely rotating about a fixed point can be parameterized by a set of Euler angles with the second of these angles having a singularity and a limited range of 180°. When the rotation of the body is constrained (as in the examples shown in Figure 2), the configuration manifold of the system changes and the range of the second angle is extended so that the two angles suffice to parameterize the rotation of the rigid body. In addition, the singularity in the parameterization of the motion vanishes. We shall discuss several other examples of this situation in the forthcoming pages.

In the present paper, we exploit recent work by Casey [4,5,6,7] on Lagrange's equations of motion for systems of particles and rigid bodies subject to holonomic constraints. In Casey's work, properties of the mechanical system are used to construct a dynamic model of a single representative particle that moves on a configuration manifold embedded in a high-dimensional space. We use the induced covariant basis for the tangent space to clearly define coordinate singularities. In addition, we explain gimbal lock as an example of a phenomenon, distinct from a coordinate singularity, that arises when a system of applied forces and moments is equivalent to a generalized force vector that is orthogonal to the configuration manifold. The orthogonality of the generalized force vector implies that it has no effect on the equations of motion for the generalized coordinates.

We begin with a review of various perspectives on the Euler angle singularity and gimbal lock in Section 2. In Section 3, we provide the relevant background on the theory of the representative particle navigating the configuration manifold. The idea of an orthogonal loading is then introduced in Section 4. Finally, in Sections 5 and 6 we explore several classical examples to help make a clear distinction between coordinate singularities and gimbal lock. Returning to the example of gimbal lock from the Apollo program, we provide an explanation in Section 6.4 for the elimination of gimbal lock by the addition of an additional gimbal.

2 A Survey of Perspectives on the Euler Angle Singularity and Gimbal Lock

The Euler angle singularity and gimbal lock receive attention from a surprising variety of disciplines. Discussions on the Euler angle singularity can be found in textbooks involving classical mechanics, computer graphics, aerospace engineering, and topology. Gimbal lock tends to feature in explanations of robotic joints, gyroscopic instrumentation, and spacecraft mechanisms. In this section, we summarize some of the perspectives on the Euler angle singularity and gimbal lock that one can find in the literature.

From a computer graphics perspective, one finds the Euler angle singularity discussed in the context of seeking a unique orientation representation and constructing interpolation paths between two orientations. To solve both problems, quaternions are often used and the SLERP algorithm is employed [25]. It is common to see the singular behavior of Euler angles being termed "gimbal lock" due to some similarities between the two phenomena. For example, Ganovelli et al. [9] mount an aircraft in nested gimbals as a *visualization aid*. In this demonstration, the Euler angle singularity for the orientation of the aircraft is coincident with gimbal lock. However, the inertia of the surrounding gimbals is ignored, so genuine gimbal lock never actually occurs. As Guha [12] explains, there is an *apparent* loss of a degree of freedom at the Euler angle singularity, which is similar to gimbal lock. However, there is no loss of a *physical* degree of freedom at the Euler angle singularity. Hanson [13] gives a short summary of gimbal lock and the Euler angle singularity, maintaining that "there are basically two related, but quite distinct, phenomena that are referred to as gimbal lock."

The clearest background on gyroscopic instruments and actual gimbal lock may be found in [15]. The author therein first defines the term "degree of freedom" in the context of gyroscopic instruments. Then, an elucidating discussion of gimbal lock ensues in the discussion of stable platforms, where Machover explains why a gyro tumbles out of control at gimbal lock due to the coincidence of gimbal axes. Kane et al. [14] correctly define gimbal lock as a coincidence of gyro axis with the outer gimbal axis for a gyro mounted in two gimbals. However, they then describe it as a problem in the description of orientations of one frame relative to another. It is apparent that they have confused gimbal lock with the Euler angle singularity. Greenwood [10] defines the Euler angle singularity in terms of non-uniqueness and an inability of the Euler angle rates to capture an angular velocity in a particular direction. He notes that the occurrence of the singularity is *coincident* with gimbal lock in gyroscope suspensions, and subsequently makes the terms synonymous, which may have added to the confusion in the literature. In Shuster's authoritative review [26], he correctly describes the Euler angle singularity in terms of non-uniqueness. By taking a 3-1-3 set, he notes that at the Euler angle singularity, a rotation about the y-axis produces sudden, finite changes in the angle values. He then relates this to gimbal lock for a mechanism similar to that shown in Figure 1(a), where, if one desires to rotate the innermost body about an axis out of the plane, an instantaneous change in the orientations of the outer two gimbals must occur.

Mebius [18] makes a clear delineation between the Euler angle singularity and gimbal lock. We are in strong agreement when Mebius states: "We remark that while the gimbal lock is a mechanical phenomenon, the Euler angle singularities are of a purely mathematical nature and have nothing to do with flight mechanics per se." Baruh introduces the concept of gimbal lock [2, Chapter 7, Section 8] in the context of transmitting rotary motion from one shaft to another through a Cardan joint. He defines gimbal lock as the inability to transmit motion by an applied moment when the two shafts are perpendicular to one another. The inability of an applied loading to affect motion to a system is our preferred notion of gimbal lock.

The topic of the present paper is also related to robotic joints. In particular, singularities of robotic joints in various mechanisms are treated from a purely kinematical perspective in [3]. In an interesting example at the conclusion of [3, Chapter 2], the authors show how an adjustment of problem parameters leads to a change in the topology of the configuration manifold which removes certain configurational singularities. In the current paper, we aim to expound upon this perspective by examining other examples where an incorporation of additional problem parameters alters the topology of the configuration manifold and has the effect of eliminating coordinate singularities.

3 Background

The systems of interest in this paper consist of particles and rigid bodies subject to integrable (or holonomic) constraints. We suppose there are p particles, R rigid bodies, and a set of H independent, integrable bilateral constraints. To establish the equations of motion for these systems, one chooses a set of $3p+6R$ curvilinear coordinates, $\left\{q^1, \ldots, q^{3p+6R}\right\}$, such that the set of H constraints can be conveniently expressed as follows:

$$q^{3p+6R-K+1} - f^K(t) = 0, \qquad (K = 1, ..., H)\,. \tag{3.1}$$

If f^K is explicitly a function of time, then constraint (3.1) is called time-dependent (or rheonomic). Otherwise, $\dot{f}^K = 0$, and constraint (3.1) is called time-independent (or scleronomic). The coordinates of interest can include Euler angles, Cartesian coordinates, and spherical polar coordinates, among others. In the sequel, we adopt the following compact notation:

$$\mathsf{q}^* = \left[q^1, \ldots, q^{3p+6R}\right]^T, \qquad \dot{\mathsf{q}}^* = \left[\dot{q}^1, \ldots, \dot{q}^{3p+6R}\right]^T, \tag{3.2}$$

where $[\cdot]^T$ denotes the transpose. To avoid confusion where it might arise, we distinguish those quantities where the integrable constraints have not been imposed with an asterisk $*$. From (3.1), we see that the constrained coordinates are indexed as $q^{N+1}, \ldots, q^{3p+6R}$. Here, N, the number of degrees of freedom of the system, is

$$N = 3p + 6R - H. \tag{3.3}$$

We also introduce the following abbreviated notations:

$$\mathsf{q} = \left[q^1, \ldots, q^N\right]^T, \qquad \dot{\mathsf{q}} = \left[\dot{q}^1, \ldots, \dot{q}^N\right]^T. \tag{3.4}$$

As illustrated in Figure 3, we denote the position vector of the ith particle by $\mathbf{x}^i$ where $i = 1, \ldots, p$, the position vector of the center of mass of the Ith rigid body by $\bar{\mathbf{x}}^I$, and the rotation tensor of the Ith rigid body that takes the reference configuration into the current configuration by $\mathbf{Q}^I$ where $I = 1, \ldots, R$.[2] For a system in the absence

[2] Additional background for a variety of sources on rotation tensors, Euler angles, Euler bases, and dual Euler bases are collected in Appendix A.

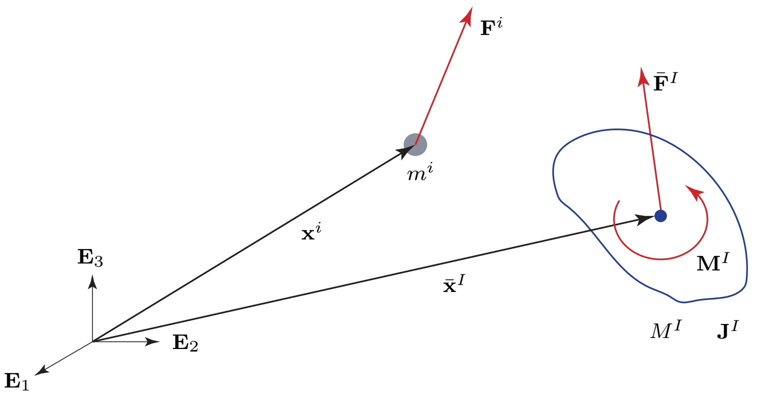

Fig. 3: Schematic of the ith particle and the Ith rigid body. A resultant force $\mathbf{F}^i$ acts on the particle of mass m^i while a resultant force $\bar{\mathbf{F}}^I$ and a resultant moment about the center of mass $\mathbf{M}^I$ acts on the Ith rigid body of mass M^I and moment of inertia tensor about the center of mass $\mathbf{J}^I$. The vectors $\mathbf{E}_1, \mathbf{E}_2$, and $\mathbf{E}_3$ form an orthonormal basis for the physical space $\mathbb{E}^3$.

of constraints, we can describe the position vector of each particle, the position vector of the center of mass of each rigid body, and the rotation tensor of each rigid body as a function of the $3p + 6R$ coordinates:

$$\begin{aligned} \mathbf{x}^i &= \mathbf{x}^i\left(\mathsf{q}^*\right), \qquad (i = 1, \dots, p), \\ \bar{\mathbf{x}}^I = \bar{\mathbf{x}}^I\left(\mathsf{q}^*\right), \qquad \mathbf{Q}^I &= \mathbf{Q}^I\left(\mathsf{q}^*\right), \qquad (I = 1, \dots, R). \end{aligned} \tag{3.5}$$

The unconstrained kinetic energy T^* of the system can be expressed as the sum of the kinetic energies of the individual particles and rigid bodies:

$$T^* = \sum_{i=1}^{p} \frac{1}{2} m^i \mathbf{v}^i \cdot \mathbf{v}^i + \sum_{I=1}^{R} \frac{1}{2}\left(M^I \bar{\mathbf{v}}^I \cdot \bar{\mathbf{v}}^I + \mathbf{J}^I \boldsymbol{\omega}^I \cdot \boldsymbol{\omega}^I\right). \tag{3.6}$$

Here, $\mathbf{v}^i = \dot{\mathbf{x}}^i$ is the velocity vector of the ith particle, m^i is the mass of the ith particle, $\bar{\mathbf{v}}^I = \dot{\bar{\mathbf{x}}}^I$ is the velocity vector of the center of mass of the Ith rigid body, M^I is the mass of the Ith rigid body, $\boldsymbol{\omega}^I$ is the angular velocity vector of the Ith rigid body, and $\mathbf{J}^I$ is the moment of inertia tensor about the center of mass of the Ith rigid body. This energy can be expressed as a function of the $3p + 6R$ coordinates and their time derivatives as

$$T^* = \frac{1}{2}\left(\dot{\mathsf{q}}^*\right)^T \mathsf{M}^* \dot{\mathsf{q}}^*, \tag{3.7}$$

where the components of the $(3p+6R) \times (3p+6R)$ symmetric matrix M^* may depend on the coordinates q^*.

When integrable constraints are imposed, the kinetic energy of the system reduces to an explicit function of the generalized coordinates, $q^1, \dots, q^N$, generalized velocities, $\dot{q}^1, \dots, \dot{q}^N$, and time, t. The kinetic energy T of the constrained system has a well-known decomposition into a term T_2 that is quadratic in the generalized velocities, a term T_1 that is linear in the generalized velocities, and a term T_0 that is independent of the generalized velocities:

$$T = T_2\left(\dot{\mathsf{q}}, \mathsf{q}, t\right) + T_1\left(\dot{\mathsf{q}}, \mathsf{q}, t\right) + T_0\left(\mathsf{q}, t\right). \tag{3.8}$$

The functions T_1 and T_0 are typically nonzero in the presence of time-dependent constraints. The quadratic term may be conveniently written as

$$T_2 = \frac{1}{2}\dot{\mathsf{q}}^T \mathsf{M} \dot{\mathsf{q}}, \tag{3.9}$$

where the $N \times N$-dimensional matrix M is known as the mass matrix. If all of the constraints are time-independent, then this matrix will not be an explicit function of time. The function T and matrix M can be obtained from T^* and M^*, respectively, by imposing the system of integrable constraints (3.1).

We next assume that a resultant force $\mathbf{F}^i$ acts on the ith particle, a resultant force $\bar{\mathbf{F}}^I$ acts at the center of mass of the Ith rigid body, and a resultant moment $\mathbf{M}^I$ acts about the center of mass of the Ith rigid body. Then, Lagrange's equations of motion for the system are

$$\frac{d}{dt}\left(\frac{\partial T^*}{\partial \dot{q}^J}\right) - \frac{\partial T^*}{\partial q^J} = Q_J, \qquad (J = 1, \ldots, 3p + 6R). \tag{3.10}$$

The generalized force Q_J is the following linear combination of the resultant forces and moments acting on the system constituents:

$$Q_J = \sum_{i=1}^{p} \mathbf{F}^i \cdot \frac{\partial \mathbf{v}^i}{\partial \dot{q}^J} + \sum_{I=1}^{R} \left(\bar{\mathbf{F}}^I \cdot \frac{\partial \bar{\mathbf{v}}^I}{\partial \dot{q}^J} + \mathbf{M}^I \cdot \frac{\partial \boldsymbol{\omega}^I}{\partial \dot{q}^J}\right). \tag{3.11}$$

The equations of motion for the integrably constrained system can often be simply found from a subset of (3.10). First, we assume that the integrable constraints are ideal. That is, the constraint forces and moments that enforce the constraints satisfy Lagrange's prescription.[3] Examples of this case include a particle that is free to move on a smooth curve or surface and a Lagrange top that is mounted on a smooth ball-and-socket joint. For the ideal case where the choice of coordinates induces constraints of the form (3.1), it can be proven (see, for example, [24]) that the constraint forces and moments acting on the system do not contribute to a subset of the Q_Js:

$$\sum_{i=1}^{p} \mathbf{F}_c^i \cdot \frac{\partial \mathbf{v}^i}{\partial \dot{q}^A} + \sum_{I=1}^{R} \left(\bar{\mathbf{F}}_c^I \cdot \frac{\partial \bar{\mathbf{v}}^I}{\partial \dot{q}^A} + \mathbf{M}_c^I \cdot \frac{\partial \boldsymbol{\omega}^I}{\partial \dot{q}^A}\right) = 0, \qquad (A = 1, \ldots, N). \tag{3.12}$$

Here, $\mathbf{F}_c^i$ is the constraint force acting on the ith particle, $\bar{\mathbf{F}}_c^I$ is the constraint force acting at the center of mass of the Ith rigid body, and $\mathbf{M}_c^I$ is the constraint moment acting at the center of mass of the Ith rigid body. In this case, Lagrange's equations of motion for the N generalized coordinates, $q^1, \ldots, q^N$, can be computed using the constrained kinetic energy (3.8):

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^A}\right) - \frac{\partial T}{\partial q^A} = Q_A, \qquad (A = 1, \ldots, N), \tag{3.13}$$

[3] In other words, the generalized constraint forces do no virtual work in any motion of the system compatible with the constraints.

and Q_A is composed solely of applied forces and moments. A canonical form of (3.13) can be found by substituting for T and expanding the partial derivatives:

$$\mathsf{M}\ddot{\mathsf{q}} + \mathsf{f}\left(\mathsf{q}, \dot{\mathsf{q}}, t\right) = \mathsf{Q}, \tag{3.14}$$

where $\mathsf{Q} = \left[Q_1, \ldots, Q_N\right]^T$ and f is an N-dimensional column array. In classic texts, such as [17,28] where the constraints are assumed to be time-independent, it is shown how the components of f can be expressed in terms of products of Christoffel symbols and generalized velocities.

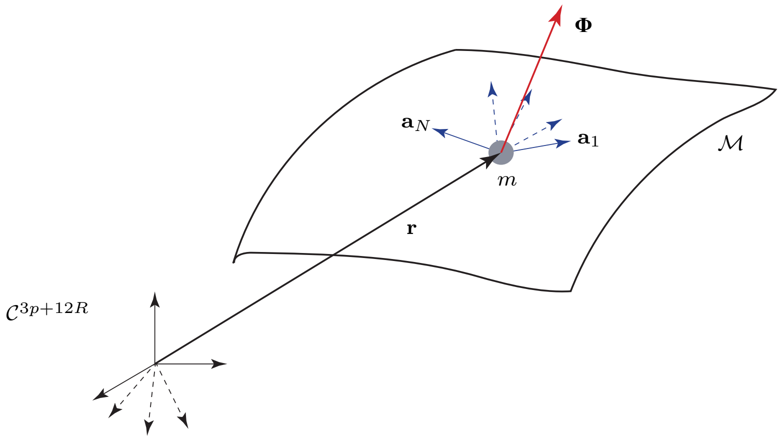

Fig. 4: Schematic of the representative particle of mass m moving on an N-dimensional configuration manifold $\mathcal{M}$. This manifold is a subset of a $(3p + 12R)$-dimensional Euclidean space $\mathcal{C}^{3p+12R}$.

Geometric interpretations of (3.10) and (3.13) can be found in the papers of Casey [4,5,6,7]. Using these works, one can construct a single representative particle of mass m moving in a $(3p + 12R)$-dimensional vector space, $\mathcal{C}^{3p+12R}$. A path in this space represents a motion of a system of p particles and differential elements of R continua. The 12 dimensions associated with each differential element result from the need for a position vector and a deformation gradient to specify a deformation of the element. By assuming rigidity of the continua in the system, the deformation gradient is a uniform proper orthogonal tensor over the whole body, and configurations are limited to a $(3p + 6R)$-dimensional subset of this space, $\mathcal{M}^*$, the configuration manifold of the unconstrained system. The position vector $\mathbf{r}$ of the representative particle and its resultant force $\mathbf{\Phi}$ are such that the equations of motion for the particle $m\ddot{\mathbf{r}} = \mathbf{\Phi}$ are equivalent to the collective balance laws for the particles and rigid bodies. In addition, the kinetic energy of the unconstrained particle is equivalent to the kinetic energy T^* of the unconstrained system: $\frac{1}{2}m\left[\dot{\mathbf{r}}, \dot{\mathbf{r}}\right] = T^*$, where $[\cdot, \cdot]$ is an inner product for $\mathcal{C}^{3p+12R}$ (see [5,7]). A single integrable constraint restricts the representative particle's motion to be in a $(3p + 6R - 1)$-dimensional hypersurface. When H independent integrable constraints are imposed on the system as in equation (3.1), the particle is constrained to move on a Riemannian N-dimensional configuration manifold $\mathcal{M}$ (cf. Figure 4) that is formed as the intersection of the H hypersurfaces. The resulting constraint force is denoted by $\boldsymbol{\Phi}_c$, which represents the collective forces and moments that manifest themselves in the presence of constraints (i.e., normal forces, reaction forces and moments, etc.).

The generalized coordinates $q^1, \ldots, q^N$ parameterize $\mathcal{M}$. The position vector of the representative particle in $\mathcal{C}^{3p+12R}$ locating a point on $\mathcal{M}^*$ is constructed as follows:

$$\mathbf{r} = \left(\mathbf{x}^1, \ldots, \mathbf{x}^p, \bar{\mathbf{x}}^1, \ldots, \bar{\mathbf{x}}^R, \mathbf{Q}^1, \ldots, \mathbf{Q}^R\right). \tag{3.15}$$

The coordinates $q^1, \ldots, q^{3p+6R}$ are then used to define a set of covariant and contravariant basis vectors in a standard manner:

$$\mathbf{a}_J = \frac{\partial \mathbf{r}}{\partial q^J}, \qquad \mathbf{a}^J = \nabla q^J, \qquad (J = 1, \ldots, 3p + 6R). \tag{3.16}$$

The subset $\{\mathbf{a}_1, \ldots \mathbf{a}_N\}$ of the covariant basis vectors span the tangent space, $T_P\mathcal{M}$ at the point $P \in \mathcal{M}$ where P indicates the point occupied by the representative particle. Most notably,

$$\begin{aligned} Q_J &= \boldsymbol{\Phi} \cdot \mathbf{a}_J \\ &= \sum_{i=1}^{p} \mathbf{F}^i \cdot \frac{\partial \mathbf{v}^i}{\partial \dot{q}^J} + \sum_{I=1}^{R} \left(\bar{\mathbf{F}}^I \cdot \frac{\partial \bar{\mathbf{v}}^I}{\partial \dot{q}^J} + \mathbf{M}^I \cdot \frac{\partial \boldsymbol{\omega}^I}{\partial \dot{q}^J} \right). \end{aligned} \tag{3.17}$$

Note that the two dot products used in equation (3.17) are defined on different spaces, the first being an inner product on $\mathcal{C}^{3p+12R}$ and the second being an ordinary dot product in Euclidean 3-space. With the help of (3.12), it may be seen that the vanishing of the constraint forces and moments from the right-hand side of Lagrange's equations (3.10) is equivalent to the constraint force $\boldsymbol{\Phi}_c$ acting on the particle of mass m being normal to $\mathcal{M}$: $\boldsymbol{\Phi}_c \cdot \mathbf{a}_A = 0$ for $A = 1, \ldots, N$.

For a mechanical system consisting of a single particle, the representative particle and the single particle are identical. In addition to this case, in the sequel, systems of a rigid body and a particle and systems of multiple rigid bodies are considered. Along with these examples, some readers may find the detailed discussion of the construction of the representative particle for systems of particles in the textbook [23] to be of interest.

4 Coordinate Singularities and Orthogonality of Loadings

The coordinate system $\left\{q^1, \ldots, q^N\right\}$ for the configuration manifold will in some cases be prone to singularities. These singularities generally manifest themselves as ambiguities in the values of $\left\{q^1, \ldots, q^N\right\}$ at some point(s) of $\mathcal{M}$. For the systems of interest in the subsequent examples, we say that a coordinate system has a singularity when the set $\{\mathbf{a}_1, \ldots \mathbf{a}_N\}$ fails to be linearly independent (and thus a basis for $T_P\mathcal{M}$) at some point $P \in \mathcal{M}$.

We note that our definition of a coordinate singularity subsumes Greenwood's definition in [10, Chapter 7, Section 13] and the notion of Euler angle singularities given in Shuster [26, Page 460]. While the kinetic energy T^* is a positive-definite quadratic function of velocity vectors $\mathbf{v}^i$, $\boldsymbol{\omega}^I$, and $\bar{\mathbf{v}}^I$, the constrained kinetic energy expressed as a function of the generalized coordinates q and their velocities $\dot{\mathsf{q}}$ may not be a positive-definite function of $\dot{\mathsf{q}}$. Specifically, at a coordinate singularity, the mass matrix M becomes singular and the kinetic energy T_2 ceases to be a positive-definite function of the velocities $\dot{\mathsf{q}}$. As M approaches the singularity, the equations of motion (3.14) become stiff and ill-posed, resulting in time integration issues with numerical schemes.

One of the main aims of this paper is to elaborate on the connections and distinctions between singularities and the notion of gimbal lock. To help achieve this goal, we now define the notion of an *orthogonal system of applied forces and moments*. Here, orthogonality pertains to the configuration manifold. We define a system of applied forces and moments such that $\mathbf{F}_a^i$ is the applied force on the ith particle, $\bar{\mathbf{F}}_a^I$ is the applied force acting at the center of mass of the Ith rigid body, and $\mathbf{M}_a^I$ is the applied moment relative to the center of mass of the Ith rigid body. The loading is said to be orthogonal to $\mathcal{M}$ if

$$\sum_{i=1}^{p} \mathbf{F}_a^i \cdot \frac{\partial \mathbf{v}^i}{\partial \dot{q}^A} + \sum_{I=1}^{R} \left(\bar{\mathbf{F}}_a^I \cdot \frac{\partial \bar{\mathbf{v}}^I}{\partial \dot{q}^A} + \mathbf{M}_a^I \cdot \frac{\partial \boldsymbol{\omega}^I}{\partial \dot{q}^A} \right) = 0, \qquad (A = 1, \ldots, N) . \tag{4.1}$$

Observe that this condition is equivalent to the corresponding active force vector, which we denote by $\mathbf{\Phi}_a$, acting on the representative particle of mass m as having vanishing covariant components:

$$\mathbf{\Phi}_a \cdot \mathbf{a}_A = 0, \qquad (A = 1, \ldots, N) . \tag{4.2}$$

Our definition of an orthogonal system of applied forces and moments presumes that there is no coordinate singularity in the covariant basis at the relevant point on the configuration manifold.

After inspecting the right-hand side of (3.13), it should be noted that such an orthogonal system of forces and moments will have no effect on the equations of motion (3.13) for the generalized coordinates of the system. If the constraints on a system are ideal and integrable, then the system of constraint forces and moments acting on the system satisfy (3.12). In the sequel, we shall find that gimbal lock is an example of an instance where an applied loading is orthogonal to the configuration manifold and thus has no effect on the dynamics of the generalized coordinates.

5 Singularities in the Dynamics of a Single Particle

In this section, we examine singularities in the spherical polar and cylindrical polar coordinate systems. By imposing constraints and incorporating the inertia of rigid body suspensions, we show how the coordinates become non-singular on new configuration manifolds. The problem of a particle sliding on a rigid half-circular hoop is treated in [23, Exercise 3.7] while a discussion of a particle on a fully circular hoop may be found in [11, Example 2.5] and [16, Chapter 2, Section 8]. However, the preceding references do not include the inertia of the supporting hoop which has interesting consequences.

5.1 The Spherical Polar Coordinate System

The dynamics of a single particle free to move in physical space are the same as the dynamics for the representative particle in the abstract space. We have the following equivalences: $\mathcal{C}^3 = \mathbb{E}^3$, $\mathcal{M}^* = \mathcal{M} = \mathcal{E}^3$, $m = m^1$, $\mathbf{r} = \mathbf{x}^1$, $\mathbf{a}_i = \frac{\partial \mathbf{v}^1}{\partial \dot{q}^i}$, and $\mathbf{\Phi} = \mathbf{F}^1$. We denote $\mathcal{E}^n$ as an n-dimensional Euclidean (flat) manifold while $\mathbb{E}^n$ denotes a Euclidean (equipped with an inner product) vector space. In spherical polar coordinates, the position and velocity vectors of both the physical particle and the representative particle are

$$\mathbf{r} = R\mathbf{e}_R, \qquad \mathbf{v} = \dot{R}\mathbf{e}_R + \dot{\theta} R \sin(\phi)\,\mathbf{e}_\theta + \dot{\phi} R \mathbf{e}_\phi, \tag{5.1}$$

where $\{R, \phi, \theta\}$ are spherical polar coordinates for $\mathcal{E}^3$, $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ is an orthonormal fixed triad for $\mathbb{E}^3$, $\mathbf{e}_r = \cos(\theta)\,\mathbf{E}_1 + \sin(\theta)\,\mathbf{E}_2$, $\mathbf{e}_\theta = \mathbf{E}_3 \times \mathbf{e}_r$, $\mathbf{e}_R = \cos(\phi)\,\mathbf{E}_3 + \sin(\phi)\,\mathbf{e}_r$, and $\mathbf{e}_\phi = \mathbf{e}_\theta \times \mathbf{e}_R$. By making the restrictions $R \in [0, \infty)$, $\theta \in [0, 2\pi)$, and $\phi \in [0, \pi]$, the parametrization is one-to-one everywhere except at $R = 0$, where θ and ϕ are arbitrary and at $\phi = 0, \pi$, where θ is arbitrary. The covariant basis is

$$\mathbf{a}_1 = \mathbf{e}_R, \qquad \mathbf{a}_2 = R\sin(\phi)\,\mathbf{e}_\theta, \qquad \mathbf{a}_3 = R\mathbf{e}_\phi. \tag{5.2}$$

The set $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ fails to be a basis for $T_P\mathcal{E}^3 = \mathbb{E}^3$ at $R = 0$ and $\phi = 0, \pi$. By our definition, the coordinate system is singular along the z-axis. The kinetic energy has the representation

$$T^* = \frac{1}{2} m \left(\dot{R}^2 + \dot{\theta}^2 R^2 \sin^2(\phi) + \dot{\phi}^2 R^2 \right). \tag{5.3}$$

T^* degenerates to a positive-semidefinite quadratic form at singular points. As a result, the induced kinematical line element for the configuration manifold, $ds^2 = \frac{2T}{m} dt^2$, is also poorly defined. The equations of motion are

$$m \begin{bmatrix} 1 & 0 & 0 \\ 0 & R^2 \sin^2(\phi) & 0 \\ 0 & 0 & R^2 \end{bmatrix} \begin{bmatrix} \ddot{R} \\ \ddot{\theta} \\ \ddot{\phi} \end{bmatrix} + m \begin{bmatrix} \dot{\theta}^2 \left(-R\sin^2(\phi)\right) + \dot{\phi}^2 (-R) \\ 2\dot{R}\dot{\theta}\left(R\sin^2(\phi)\right) + 2\dot{\theta}\dot{\phi}\left(R^2 \sin(\phi)\cos(\phi)\right) \\ \dot{\theta}^2 \left(-R^2 \sin(\phi)\cos(\phi)\right) + 2\dot{R}\dot{\phi}(R) \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{\Phi} \cdot \mathbf{e}_R \\ \mathbf{\Phi} \cdot R\sin(\phi)\,\mathbf{e}_\theta \\ \mathbf{\Phi} \cdot R\mathbf{e}_\phi \end{bmatrix}. \tag{5.4}$$

At the singularity, we observe that the mass matrix is singular. The non-invertibility of M presents an issue in the numerical integration of the equations of motion, where the coordinate accelerations become infinite. If a nonzero force $\mathbf{F}^1$ is applied along the z-axis in the $\mathbf{e}_\theta$ direction, it goes undetected by the equations of motion. We emphasize that this does not correspond to $\mathbf{\Phi}$ being orthogonal to $\mathcal{M}$. The component of force is not captured due to singularities in the coordinate system. $\mathbf{F}^1$ *does* affect motion to the system. One would be mistaken to refer to this degenerate behavior as "gimbal lock."

5.2 A Particle on a Rotating Half-Line

Any ambiguities in θ or ϕ are avoided by explicitly prescribing their value for all time as follows:

$$\theta - f^1(t) = 0, \quad \phi - f^2(t) = 0. \tag{5.5}$$

The corresponding constraints on the velocity vector are

$$\mathbf{v} \cdot \frac{1}{R\sin(f^2)} \mathbf{e}_\theta - \dot{f}^1 = 0, \quad \mathbf{v} \cdot \frac{1}{R} \mathbf{e}_\phi - \dot{f}^2 = 0. \tag{5.6}$$

The resulting constrained velocity vector is

$$\mathbf{v} = \dot{R}\mathbf{e}_R + \dot{f}^1 R \sin\left(f^2\right)\mathbf{e}_\theta + \dot{f}^2 R \mathbf{e}_\phi. \tag{5.7}$$

With $R \in [0, \infty)$ inherited from the parametrization for $\mathcal{E}^3$, the imposition of these constraints yields $\mathcal{M} = H^1$, a half-line issuing from the origin with angular velocity $\dot{f}^1\mathbf{E}_3 + \dot{f}^2\mathbf{e}_\theta$. The covariant vector

$$\mathbf{a}_1 = \mathbf{e}_R \tag{5.8}$$

provides a well-defined basis for $T_P H^1 = \mathbb{E}^1$, so there are no coordinate singularities. The relative velocity vector $\mathbf{v} - \frac{\partial \mathbf{r}}{\partial t} = \dot{R}\mathbf{e}_R$ is an element of this space. Comparing the bases in equations (5.2) and (5.8), we see that we have eliminated the problematic vectors through the imposition of constraints. Lagrange's prescription gives the required constraint force as

$$\mathbf{\Phi}_c = \mu_1 \frac{1}{R\sin\left(f^2\right)}\mathbf{e}_\theta + \mu_2 \frac{1}{R}\mathbf{e}_\phi. \tag{5.9}$$

Since R came from the ambient space, we must enforce the unilateral constraint $R \geq 0$, which is of the form

$$\mathbf{v}\cdot\mathbf{e}_R \geq 0 \quad \text{at} \quad R = 0. \tag{5.10}$$

The corresponding constraint impulse required at $R = 0$ has the prescription

$$\hat{\mathbf{\Phi}}_c = m\mathbf{v}^+ - m\mathbf{v}^-. \tag{5.11}$$

The constrained kinetic energy is

$$T = T_2 + T_0 = \frac{1}{2}m\dot{R}^2 + \frac{1}{2}m\left(\dot{f}^1\dot{f}^1R^2\sin^2\left(f^2\right) + \dot{f}^2\dot{f}^2R^2\right) \tag{5.12}$$

The kinetic energy capturing movement relative to the configuration manifold, T_2, is clearly a non-degenerate quadratic form of $\dot{R}$. The sole equation of motion and constraint force is

$$m\ddot{R} - mR\left(\dot{f}^2\dot{f}^2 + \dot{f}^1\dot{f}^1\sin^2\left(f^2\right)\right) = \mathbf{\Phi}_a\cdot\mathbf{e}_R, \tag{5.13}$$

with

$$\begin{aligned}\mathbf{\Phi}_c = &\left(m\ddot{f}^1R\sin\left(f^2\right) + 2m\dot{f}^2\left(\dot{R}\sin\left(f^2\right) + \dot{f}^2R\cos\left(f^2\right)\right) - \mathbf{\Phi}_a\cdot\mathbf{e}_\theta\right)\mathbf{e}_\theta \\ &+ \left(m\ddot{f}^2R + 2m\dot{R}\dot{f}^2 - m\dot{f}^1\dot{f}^1R\sin\left(f^2\right)\cos\left(f^2\right) - \mathbf{\Phi}_a\cdot\mathbf{e}_\phi\right)\mathbf{e}_\phi.\end{aligned} \tag{5.14}$$

The mass matrix in equation (5.13) is simply m and is invertible as a nonzero scalar. To describe a particle on a full line with configuration manifold $\mathcal{E}^1$, an extension of R from the range $[0, \infty)$ to $R_e \in (-\infty, \infty)$ is required. The resulting equation of motion is identical to equation (5.13) with constraint force identical to equation (5.14) but no constraint impulse $\hat{\mathbf{\Phi}}_c$ is required.

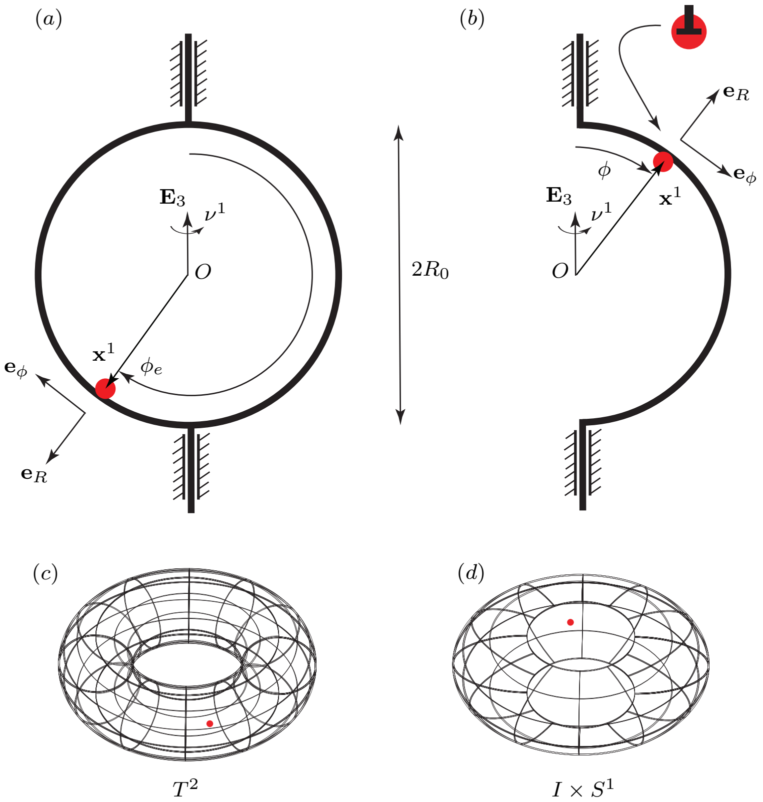

Fig. 5: *A particle sliding on a (a) circular hoop and a (b) half-circular hoop and their respective configuration manifolds, (c)* T^2 *and (d)* $I \times S^1$.

5.3 The Effect of an Additional Inertia

Consider a mechanical system consisting of a particle sliding on a rigid circular hoop of radius R_0 that is engaged in fixed-axis rotation about $\mathbf{E}_3$, as shown in Figure 5(b). Let $\{x_1, x_2, x_3\}$ be rectangular Cartesian coordinates for $\bar{\mathbf{x}}^1$, $\{R, \theta, \phi_e\}$ be spherical polar coordinates for $\mathbf{x}^1 - \bar{\mathbf{x}}^1$, and $\left\{\nu^1, \nu^2, \nu^3\right\}$ be a set of 3-2-1 Euler angles for $\mathbf{Q}^1$. Here, ϕ_e is the extension of the ordinary polar angle ϕ, so that $\phi_e \in [0, 2\pi)$. The reference configuration of the hoop is such that a normal to its face is in the $\mathbf{E}_2$-direction. A corotational basis $\left\{\mathbf{e}_i{}^1\right\}$ is attached to the hoop for all time. For the unconstrained system, we have the following kinematical quantities:

$$\bar{\mathbf{x}}^1 = \sum_{i=1}^{3} x_i \mathbf{E}_i, \qquad \mathbf{x}^1 - \bar{\mathbf{x}}^1 = R\mathbf{e}_R, \tag{5.15}$$

and the components of $\mathbf{Q}^1$ are gotten from equation (A.7). Note that the set $\left\{\mathbf{e}_R, \mathbf{e}_\theta, \mathbf{e}_\phi\right\}$ is relative to the corotational basis $\left\{\mathbf{e}_i{}^1\right\}$. The position vector of the representative particle in $\mathcal{C}^{15}$ is

$$\mathbf{r} = \left(\mathbf{x}^1, \bar{\mathbf{x}}^1, \mathbf{Q}^1\right). \tag{5.16}$$

The unconstrained velocities in the physical system are

$$\bar{\mathbf{v}}^1 = \sum_{i=1}^{3} \dot{x}_i \mathbf{E}_i, \qquad \boldsymbol{\omega}^1 = \sum_{i=1}^{3} \dot{\nu}^i \mathbf{g}_i,$$
$$\mathbf{v}^1 - \bar{\mathbf{v}}^1 = \dot{R}\mathbf{e}_R + \dot{\theta} R \sin(\phi_e)\, \mathbf{e}_\theta + \dot{\phi}_e R \mathbf{e}_\phi + \boldsymbol{\omega}^1 \times \left(\mathbf{x}^1 - \bar{\mathbf{x}}^1\right). \tag{5.17}$$

The generalized coordinates for the system are $\left\{q^1, q^2\right\} = \left\{\nu^1, \phi_e\right\}$ while the constrained coordinates are $\left\{q^3, \ldots, q^9\right\} = \left\{x_1, x_2, x_3, \nu^2, \nu^3, R, \theta\right\}$. Before the constraints are imposed, we have the following sets:

$$\left\{\frac{\partial \mathbf{v}^1}{\partial \dot{q}^J}\right\} = \left\{\mathbf{g}_1 \times \mathbf{d}, R\mathbf{e}_\phi, \mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3, \mathbf{g}_2 \times \mathbf{d}, \mathbf{g}_3 \times \mathbf{d}, \mathbf{e}_R, R\sin(\phi_e)\, \mathbf{e}_\theta\right\},$$
$$\left\{\frac{\partial \bar{\mathbf{v}}^1}{\partial \dot{q}^J}\right\} = \left\{\mathbf{0}, \mathbf{0}, \mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}\right\}, \qquad \left\{\frac{\partial \boldsymbol{\omega}^1}{\partial \dot{q}^J}\right\} = \left\{\mathbf{g}_1, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{g}_2, \mathbf{g}_3, \mathbf{0}, \mathbf{0}\right\}. \tag{5.18}$$

where $\mathbf{d} = \mathbf{x}^1 - \bar{\mathbf{x}}^1$. The seven integrable constraints are

$$x_1 = 0, \quad x_2 = 0, \quad x_3 = 0, \quad \nu^2 = 0, \quad \nu^3 = 0, \quad R - R_0 = 0, \quad \theta = 0. \tag{5.19}$$

In terms of velocity vectors, we have

$$\bar{\mathbf{v}}^1 \cdot \mathbf{E}_1 = 0, \quad \bar{\mathbf{v}}^1 \cdot \mathbf{E}_2 = 0, \quad \bar{\mathbf{v}}^1 \cdot \mathbf{E}_3 = 0, \quad \boldsymbol{\omega}^1 \cdot \mathbf{e}_\theta = 0, \quad \boldsymbol{\omega}^1 \cdot \mathbf{e}_r = 0,$$
$$\mathbf{v}^1 \cdot \mathbf{e}_R - \bar{\mathbf{v}}^1 \cdot \mathbf{e}_R = 0, \quad \mathbf{v}^1 \cdot \mathbf{e}_\theta - \bar{\mathbf{v}}^1 \cdot \mathbf{e}_\theta + \boldsymbol{\omega}^1 \cdot R_0 \mathbf{e}_\phi = 0, \tag{5.20}$$

where some of the constraints have been applied to simplify other constraint expressions, for example: $\boldsymbol{\omega}^1 \cdot \mathbf{g}^2 = \boldsymbol{\omega}^1 \cdot \mathbf{e}_\theta$ (see Appendix A for background on the dual Euler basis $\{\mathbf{g}^i\}$). After the constraints are imposed, $\mathcal{M} = T^2$, the 2-torus (Figure 5(c)). Lagrange's prescription yields the constraint forces and moments:

$$\mathbf{F}_c^1 = \mu_6 \mathbf{e}_R + \mu_7 \mathbf{e}_\theta, \qquad \mathbf{M}_c^1 = \mu_4 \mathbf{e}_\theta + \mu_5 \mathbf{e}_r + \mu_7 R_0 \mathbf{e}_\phi,$$
$$\bar{\mathbf{F}}_c^1 = \mu_1 \mathbf{E}_1 + \mu_2 \mathbf{E}_2 + \mu_3 \mathbf{E}_3 - \mu_6 \mathbf{e}_R - \mu_7 \mathbf{e}_\theta. \tag{5.21}$$

The generalized forces are

$$Q_1 = \mathbf{F}_a^1 \cdot R_0 \sin(\phi_e)\, \mathbf{e}_\theta + \mathbf{M}_a^1 \cdot \mathbf{E}_3, \qquad Q_2 = \mathbf{F}_a^1 \cdot R_0 \mathbf{e}_\phi,$$
$$Q_{(2+i)} = \mathbf{F}_a^1 \cdot \mathbf{E}_i + \bar{\mathbf{F}}_a^1 \cdot \mathbf{E}_i + \mu_i, \qquad (i = 1, 2, 3),$$
$$Q_6 = \mathbf{M}_a^1 \cdot \mathbf{e}_\theta + \mu_4, \qquad Q_7 = \mathbf{F}_a^1 \cdot (-R_0 \cos(\phi_e)\, \mathbf{e}_\theta) + \mathbf{M}_a^1 \cdot \mathbf{e}_r + \mu_5,$$
$$Q_8 = \mathbf{F}_a^1 \cdot \mathbf{e}_R + \mu_6, \qquad Q_9 = \mathbf{F}_a^1 \cdot R_0 \sin(\phi_e)\, \mathbf{e}_\theta + \mu_7 R_0 \sin(\phi_e). \tag{5.22}$$

In the abstract setting, the constraint forces and moments are thought to ensure that the representative particle remains on T^2. Being interested in only the equations of motion and not equations for generalized constraint forces, we may impose the constraints on the velocities:

$$\mathbf{v}^1 = \dot{\nu}^1 R_0 \sin(\phi_e)\, \mathbf{e}_\theta + \dot{\phi}_e R_0 \mathbf{e}_\phi, \quad \bar{\mathbf{v}}^1 = \mathbf{0}, \quad \boldsymbol{\omega}^1 = \dot{\nu}^1 \mathbf{E}_3. \tag{5.23}$$

We have a covariant basis for T_PT^2 as

$$\mathbf{a}_1 = \frac{\partial \mathbf{r}}{\partial \nu^1} = \left(R_0 \sin\left(\phi_e\right) \mathbf{e}_\theta, \mathbf{0}, \operatorname{skwt}\left(\mathbf{E}_3\right) \mathbf{Q}^1\right),$$
$$\mathbf{a}_2 = \frac{\partial \mathbf{r}}{\partial \phi_e} = \left(R_0 \mathbf{e}_\phi, \mathbf{0}, \mathbf{O}\right), \tag{5.24}$$

where $\mathbf{O}$ is the zero-tensor for the 9-dimensional vector space of linear second-order tensors and the linear map skwt $(\cdot)$ is defined in Appendix A. The covariant basis spans a 2-dimensional vector space for all values of $\mathsf{q} = \left[\nu^1, \phi_e\right]^T$. Therefore, the coordinate system is singularity-free. If we let the hoop have a radius of gyration k about $\mathbf{E}_3$ through O, we have the following expression for the constrained kinetic energy of the physical system and the representative particle:

$$T = \frac{1}{2} m^1 \left(\left(\dot{\nu}^1\right)^2 R_0^2 \sin^2\left(\phi_e\right) + \dot{\phi}_e^2 R_0^2 \right) + \frac{1}{2} M^1 k^2 \left(\dot{\nu}^1\right)^2, \tag{5.25}$$

which is a non-degenerate quadratic form for all q. The equations of motion are

$$\begin{bmatrix} m^1 R_0^2 \sin^2\left(\phi_e\right) + M^1 k^2 & 0 \\ 0 & m^1 R_0^2 \end{bmatrix} \begin{bmatrix} \ddot{\nu}^1 \\ \ddot{\phi}_e \end{bmatrix} + m^1 \begin{bmatrix} 2\dot{\nu}^1 \dot{\phi}_e \left(R_0^2 \sin\left(\phi_e\right)\cos\left(\phi_e\right)\right) \\ \left(\dot{\nu}^1\right)^2 \left(-R_0^2 \sin\left(\phi_e\right)\cos\left(\phi_e\right)\right) \end{bmatrix} = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}. \tag{5.26}$$

The presence of the moment of inertia M^1k^2 makes the mass matrix (and the kinetic energy) non-singular. The quantity M^1k^2 may be thought to control the outer radius of the configuration manifold, T^2. When $M^1k^2 \to 0$, T^2 degenerates into S^2, the 2-sphere, and $\left\{\nu^1, \phi_e\right\}$ become the usual extended spherical polar coordinates on S^2, replete with singularities at $\phi_e = 0, \pi$. By including the effects of hoop inertia in the dynamics, we have generated a configuration manifold on which the coordinates have no singularities.

At $\phi_e = 0, \pi$, the particle comes into line with the axis of rotation of the hoop. When at this location, a nonzero force applied to the particle in $\mathbf{e}_\theta$ affects no motion to the system: it does not contribute to Q_1 or Q_2. Instead, it gets absorbed into $Q_3, \ldots, Q_9$ and is counteracted entirely by constraint forces and moments. There are two circles in T^2 where the system experiences a phenomenon that one might call "gimbal lock." This behavior is similar to the classic notion of gimbal lock, where a gyro axis comes into alignment with a gimbal axis. At a point anywhere on one of the circles in T^2, if $\mathbf{F}_a^1 = F_a^1 \mathbf{e}_\theta$, $F_a^1 \neq 0$, then $\mathbf{\Phi}_a$ is orthogonal to T^2.

To describe a particle on a half-circular hoop, as shown in Figure 5(b), we retain the original range of the polar coordinate: $\phi \in [0, \pi]$. Since O is a fixed point, the equations of motion are identical to (5.26), but with $\phi_e \mapsto \phi$ and all forces and moments are relative to point O. Two unilateral constraints $\phi \geq 0$ and $\phi \leq 0$ are imposed to produce a configuration manifold with two boundaries. In this case, it is the cylinder, $I \times S^1$ (Figure 5(d)). We require two constraint impulses applied to the particle at $\phi = 0$ and $\phi = \pi$ to enforce the two unilateral constraints. An extension of ϕ into ϕ_e, a cyclic coordinate, corresponds to gluing the boundaries of $I \times S^1$ together, resulting in T^2, the configuration manifold for a fully circular hoop.

The effect of unilateral constraints on the phase portrait of a particle on a hoop controlled to rotate at a constant rate $\dot{\nu}^1 = \mathit{\Omega}$ is shown in Figure 6. The equations of motion for the particle on the full hoop and the half-hoop under the influence of gravitational acceleration g are

$$\ddot{\phi}_e = \left(\Omega^2 \cos\left(\phi_e\right) + \frac{g}{R_0}\right) \sin\left(\phi_e\right),$$
$$\ddot{\phi} = \left(\Omega^2 \cos\left(\phi\right) + \frac{g}{R_0}\right) \sin\left(\phi\right), \tag{5.27}$$

respectively. For the half-hoop, we assume an instantaneous perfectly elastic impact at $\phi = 0, \pi$ and that the impulsive force resulting from the impact is far more significant than any other forces applied to the particle. Then,

$$\dot{\phi}^{+} = -\dot{\phi}^{-} \quad \text{at } \phi = 0 \text{ and } \phi = \pi, \tag{5.28}$$

where $\dot{\phi}^{-}$ and $\dot{\phi}^{+}$ are the values of $\dot{\phi}$ immediately before and after the collision, respectively.

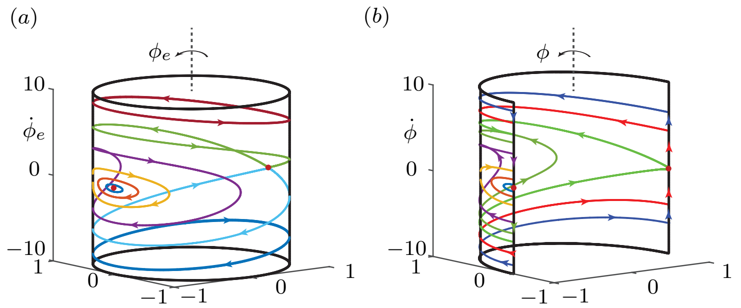

Fig. 6: *Phase portrait of solutions to (a) equation (5.27)$_1$ plotted on $I \times S^1$ and (b) equation (5.27)$_2$ plotted on $I \times I$.*

5.4 An Analogous Problem: Cylindrical Polar Coordinates

A similar problem involves using cylindrical polar coordinates to describe $\mathcal{E}^3$, which also suffer from singularities along the z-axis, where θ is arbitrary. The covariant basis

$$\mathbf{a}_1 = \mathbf{e}_r, \qquad \mathbf{a}_2 = r\mathbf{e}_\theta, \qquad \mathbf{a}_3 = \mathbf{E}_3, \tag{5.29}$$

clearly fails to span $T_P\mathcal{E}^3 = \mathbb{E}^3$ at $r = 0$. As detailed in Section 5.1, implications of the singularity include a degenerate kinetic energy and singular equations of motion.

We can eliminate the ambiguity in θ by explicitly defining its value through the constraint

$$\theta - f\left(t\right) = 0. \tag{5.30}$$

Using the constrained velocity vector

$$\mathbf{v} = \dot{r}\mathbf{a}_1 + \dot{z}\mathbf{a}_2 + \frac{\partial \mathbf{r}}{\partial t} = \dot{r}\mathbf{e}_r + \dot{z}\mathbf{E}_3 + \dot{f}r\mathbf{e}_\theta, \tag{5.31}$$

the covariant basis vectors are easily read off:

$$\mathbf{a}_1 = \frac{\partial \mathbf{v}}{\partial \dot{r}} = \mathbf{e}_r, \qquad \mathbf{a}_2 = \frac{\partial \mathbf{v}}{\partial \dot{z}} = \mathbf{E}_3. \tag{5.32}$$

The set $\{\mathbf{a}_1, \mathbf{a}_2\}$ is everywhere a well-defined basis for describing $\mathbf{v} - \frac{\partial \mathbf{r}}{\partial t}$. If the radial coordinate is extended to $r_e \in (-\infty, \infty)$, then the configuration manifold is a rotating plane, $\mathcal{E}^2$. If instead the radial coordinate is inherited from the ambient space, then the configuration manifold is a rotating half-plane, H^2, and the unilateral constraint $r \leq 0$ must be imposed. Either way, $\{r_e, z\}$ and $\{r, z\}$ yield non-degenerate coordinatizations of $\mathcal{M}$.

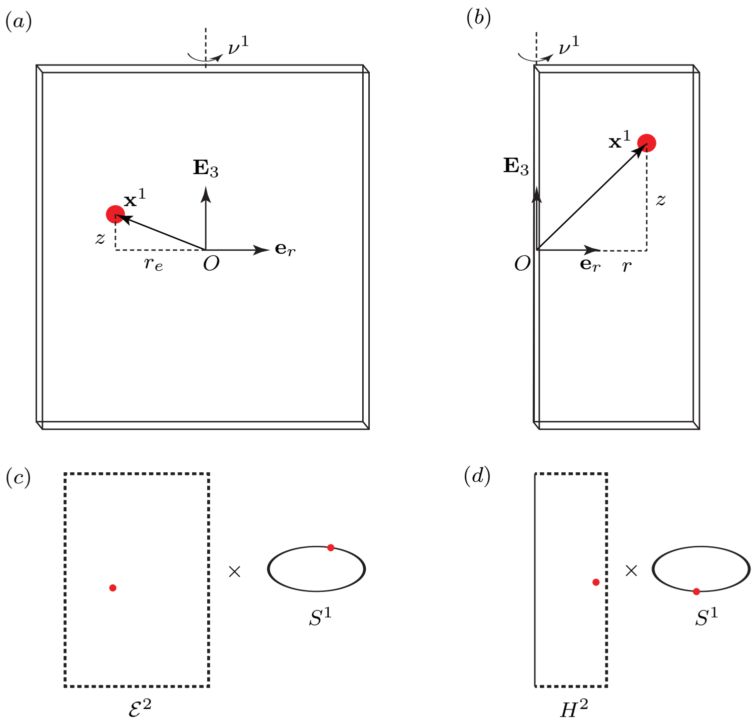

Fig. 7: *A particle sliding between two (a) full plates and (b) half-plates and their respective configuration manifolds, (c)* $\mathcal{E}^2 \times S^1$ *and (d)* $H^2 \times S^1$.

Additionally, the singularity is eliminated if we consider the problem of a particle confined to move between two rotating plates. By including the plate inertias, the configuration manifold becomes either $\mathcal{E}^2 \times S^1$ or $H^2 \times S^1$, objects which are well-parametrized by $\{r_e, \nu^1, z\}$ and $\{r, \nu^1, z\}$, respectively (Figure 7). The covariant basis for the 3-dimensional tangent space $T_P\left(\mathcal{E}^2 \times S^1\right)$ is found to be:

$$\mathbf{a}_1 = \frac{\partial \mathbf{r}}{\partial r_e} = (\mathbf{e}_r, \mathbf{0}, \mathbf{O}), \qquad \mathbf{a}_2 = \frac{\partial \mathbf{r}}{\partial \nu^1} = \left(r_e \mathbf{e}_\theta, \mathbf{0}, \mathsf{skwt}\,(\mathbf{E}_3)\,\mathbf{Q}^1\right),$$
$$\mathbf{a}_3 = \frac{\partial \mathbf{r}}{\partial z} = (\mathbf{E}_3, \mathbf{0}, \mathbf{O}). \tag{5.33}$$

This set is orthogonal everywhere in $\mathcal{E}^2 \times S^1$. Letting the plates together possess a finite radius of gyration k about $\mathbf{E}_3$ through O, one also finds the following non-degenerate kinetic energy:

$$T = T_2 = \frac{1}{2} m^1 \left(\dot{r}_e^2 + \dot{z}^2 + \left(\dot{\nu}^1\right)^2 r_e^2 \right) + \frac{1}{2} M^1 k^2 \left(\dot{\nu}^1\right)^2 . \tag{5.34}$$

Finally, the equations of motion are formulated as

$$\begin{bmatrix} m^1 & 0 & 0 \\ 0 & m^1 r_e^2 + M^1 k^2 & 0 \\ 0 & 0 & m^1 \end{bmatrix} \begin{bmatrix} \ddot{r}_e \\ \ddot{\nu}^1 \\ \ddot{z} \end{bmatrix} + m^1 \begin{bmatrix} \left(\dot{\nu}^1\right)^2 (-r_e) \\ 2\dot{r}_e \dot{\nu}^1 (r_e) \\ 0 \end{bmatrix} = \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix}, \tag{5.35}$$

where

$$Q_1 = \mathbf{F}_a^1 \cdot \mathbf{e}_r, \qquad Q_2 = \mathbf{F}_a^1 \cdot r_e \mathbf{e}_\theta + \mathbf{M}_a^1 \cdot \mathbf{E}_3, \qquad Q_3 = \mathbf{F}_a^1 \cdot \mathbf{E}_3. \tag{5.36}$$

The mass matrix is only singular when $Mk^2 = 0$. The quantity Mk^2 may be thought to control the radius of S^1. When it tends to zero, the configuration manifold collapses back into $\mathcal{E}^3$ and the singularities reappear. This mechanical system is subject to gimbal lock when $r_e = 0$ or $r = 0$, where the only orthogonal loading with nonzero $\mathbf{F}_a^1$ may be found.

6 Singularities in the Dynamics of a Rigid Body

It has been shown that no global three-parameter representation of the rotation group exists without singularities [27], and the Euler angles are no exception. In this section, we explore the Euler angle singularity and see how it is eliminated through the imposition of constraints and the addition of system inertias. We will also demonstrate that gimbal lock corresponds to an applied loading that is orthogonal to the configuration manifold and how an additional constrained gimbal may be added to avoid gimbal lock.

6.1 The Euler Angle Coordinate System

Suppose a body is constrained so that its center of mass is a fixed point. Then, $\bar{\mathbf{x}}^1 = \mathbf{0}$ and the body engages in fixed-point rotation about its center of mass. The position vector to the representative particle in $\mathcal{C}^{12}$ is then

$$\mathbf{r} = \left(\mathbf{0}, \mathbf{Q}^1\right), \tag{6.1}$$

where $\mathbf{Q}^1$ is the rotation tensor of the body. The position vector $\mathbf{r}$ locates a point in $\mathcal{M} = \mathbb{R}P^3$, where $\mathbb{R}P^3$ is real projective 3-space, and the Euler angles $\left\{\nu^1, \nu^2, \nu^3\right\}$ provide a parametrization of this space. The tangent space to $\mathbb{R}P^3$ is a rotation of Skw by $\mathbf{Q}^1$. A covariant basis for the tangent space is obtained as

$$\mathbf{a}_1 = \frac{\partial \mathbf{r}}{\partial \nu^1} = \left(\mathbf{0}, \operatorname{skwt}\left(\mathbf{g}_1\right) \mathbf{Q}^1\right), \qquad \mathbf{a}_2 = \frac{\partial \mathbf{r}}{\partial \nu^2} = \left(\mathbf{0}, \operatorname{skwt}\left(\mathbf{g}_2\right) \mathbf{Q}^1\right),$$
$$\mathbf{a}_3 = \frac{\partial \mathbf{r}}{\partial \nu^3} = \left(\mathbf{0}, \operatorname{skwt}\left(\mathbf{g}_3\right) \mathbf{Q}^1\right). \tag{6.2}$$

For every Euler angle parametrization there are two values of ν^2 for which $\mathbf{g}_1 = \mathbf{g}_3$ and $\mathbf{g}_1 = -\mathbf{g}_3$. For example, the Euler basis for the 3-1-3 set is expressed in linear combinations of the inertial basis as

$$\begin{bmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \\ \mathbf{g}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ \cos\left(\nu^1\right) & \sin\left(\nu^1\right) & 0 \\ \sin\left(\nu^2\right)\sin\left(\nu^1\right) & -\sin\left(\nu^2\right)\cos\left(\nu^1\right) & \cos\left(\nu^2\right) \end{bmatrix} \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}. \tag{6.3}$$

At $\nu^2 = 0$, $\mathbf{g}_1 = \mathbf{g}_3$ and at $\nu^2 = \pi$, $\mathbf{g}_1 = -\mathbf{g}_3$. One can also show by substituting these critical values into expression (A.8) that there are ambiquities in the value of $\nu^1 + \nu^3$ or $\nu^1 - \nu^3$. An important consequence that occurs when $\mathbf{g}_1 = \pm\mathbf{g}_3$ is the failure of the set $\{\text{skwt}(\mathbf{g}_1), \text{skwt}(\mathbf{g}_2), \text{skwt}(\mathbf{g}_3)\}$ to span Skw, which leads to $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ failing to span $T_P\mathbb{R}P^3$, indicating that the Euler angle parametrization is corrupted by singularities at points on $\mathbb{R}P^3$. Using the isomorphism between Skw and $\mathbb{E}^3$, it is equivalent to say that the Euler basis $\{\mathbf{g}_i\}$ fails to capture every $\boldsymbol{\omega}^1 \in \mathbb{E}^3$ at the Euler angle singularity.

To illustrate the effect of the singularity on the parametrization of the kinetic energy and the equations of motion, consider a homogeneous spherically symmetric body with moment of inertia tensor $\mathbf{J}^1 = \lambda^1\mathbf{I}$ about its center of mass. The constrained kinetic energy of the body is

$$T = \frac{\lambda^1}{2}\left(\left(\dot{\nu}^1\right)^2 + \left(\dot{\nu}^2\right)^2 + \left(\dot{\nu}^3\right)^2 + 2\dot{\nu}^1\dot{\nu}^3\cos\left(\nu^2\right)\right), \tag{6.4}$$

which fails to be a positive-definite quadratic form in $\dot{\mathsf{q}} = \left[\dot{\nu}^1, \dot{\nu}^2, \dot{\nu}^3\right]^T$ at $\nu^2 = 0, \pi$. The equations of motion are

$$\lambda^1 \begin{bmatrix} 1 & 0 & \cos\left(\nu^2\right) \\ 0 & 1 & 0 \\ \cos\left(\nu^2\right) & 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{\nu}^1 \\ \ddot{\nu}^2 \\ \ddot{\nu}^3 \end{bmatrix} - \lambda^1\sin\left(\nu^2\right) \begin{bmatrix} \dot{\nu}^2\dot{\nu}^3 \\ \dot{\nu}^3\dot{\nu}^1 \\ \dot{\nu}^1\dot{\nu}^2 \end{bmatrix} = \begin{bmatrix} \mathbf{M}^1\cdot\mathbf{g}_1 \\ \mathbf{M}^1\cdot\mathbf{g}_2 \\ \mathbf{M}^1\cdot\mathbf{g}_3 \end{bmatrix}. \tag{6.5}$$

The mass matrix is non-invertible at $\nu^2 = 0, \pi$. Notice also at the singularity that a moment applied in the $\mathbf{g}_1 \times \mathbf{g}_2$-direction is not captured by the right hand side of equation (6.5). This result is due to the coordinate singularity and not due to gimbal lock. For the representative particle, we have the equivalence $Q_i = \mathbf{\Phi}\cdot\mathbf{a}_i = \mathbf{M}^1\cdot\mathbf{g}_i$, where $\mathbf{\Phi}$ is a vector in $\mathcal{C}^{12}$ involving the skew-symmetric tensor of the applied moment $\mathbf{M}^1$. For further details, see [7].

6.2 Imposition of Constraints

Suppose ν^1 is to be constrained such that

$$\nu^1 - f(t) = 0. \tag{6.6}$$

For the 3-1-3 Euler angle set, the dual Euler basis has the representation

$$\begin{bmatrix} \mathbf{g}^1 \\ \mathbf{g}^2 \\ \mathbf{g}^3 \end{bmatrix} = \begin{bmatrix} -\sin\left(\nu^1\right)\cot\left(\nu_e^2\right) & \cos\left(\nu^1\right)\cot\left(\nu_e^2\right) & 1 \\ \cos\left(\nu^1\right) & \sin\left(\nu^1\right) & 0 \\ \sin\left(\nu^1\right)\csc\left(\nu_e^2\right) & -\cos\left(\nu^1\right)\csc\left(\nu_e^2\right) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}, \tag{6.7}$$

where $\nu^2 \in [0,\pi]$ has been extended to $\nu_e^2 \in [0,2\pi)$ (cf. Section A.1). We may write constraint (6.6) in terms of the angular velocity vector as

$$\boldsymbol{\omega}^1 \cdot \mathbf{g}^1 - \dot{f} = 0. \tag{6.8}$$

Lagrange's prescription yields an expression for the required constraint moment as

$$\mathbf{M}_c^1 = \mu \mathbf{g}^1. \tag{6.9}$$

The constrained angular velocity vector is

$$\boldsymbol{\omega}^1 = \dot{\nu}_e^2 \mathbf{g}_2 + \dot{\nu}^3 \mathbf{g}_3 + \dot{f}\mathbf{g}_1. \tag{6.10}$$

The generalized coordinates in this case are $\left\{\nu_e^2, \nu^3\right\}$ and the configuration manifold is the 2-torus, T^2. A covariant basis for the tangent space $T_P T^2$ is

$$\mathbf{a}_1 = \frac{\partial \mathbf{r}}{\partial \nu_e^2} = \left(\mathbf{0}, \operatorname{skwt}\left(\mathbf{g}_2\right)\mathbf{Q}^1\right), \qquad \mathbf{a}_2 = \frac{\partial \mathbf{r}}{\partial \nu^3} = \left(\mathbf{0}, \operatorname{skwt}\left(\mathbf{g}_3\right)\mathbf{Q}^1\right). \tag{6.11}$$

Since $\mathbf{g}_2 \cdot \mathbf{g}_3 = 0$ always, the coordinate system is singularity-free. The constrained kinetic energy is given by

$$T = T_2 + T_1 + T_0 = \frac{1}{2}\lambda^1\left(\left(\dot{\nu}_e^2\right)^2 + \left(\dot{\nu}^3\right)^2\right) + \lambda^1 \dot{f}\dot{\nu}^3 \cos\left(\nu_e^2\right) + \frac{1}{2}\lambda^1 \dot{f}^2. \tag{6.12}$$

Here, T_2 is a well-defined positive-definite quadratic form in $\dot{\mathsf{q}} = \left[\dot{\nu}_e^2, \dot{\nu}^3\right]^T$ for all q. The equations of motion resulting from equation (6.5) after the constraint is imposed are

$$\lambda^1 \begin{bmatrix}\ddot{\nu}_e^2 \\ \ddot{\nu}^3\end{bmatrix} - \lambda^1 \sin\left(\nu_e^2\right)\begin{bmatrix}\dot{f}\dot{\nu}^3 \\ \dot{f}\dot{\nu}_e^2\end{bmatrix} = \begin{bmatrix}\mathbf{M}_a^1 \cdot \mathbf{g}_2 \\ \mathbf{M}_a^1 \cdot \mathbf{g}_3 - \lambda^1 \ddot{f}\cos\left(\nu_e^2\right)\end{bmatrix}, \tag{6.13}$$

with

$$\mathbf{M}_c^1 = \left(\lambda^1 \ddot{f} + \lambda^1 \ddot{\nu}^3 \cos\left(\nu_e^2\right) - \lambda^1 \dot{\nu}_e^2 \dot{\nu}^3 \sin\left(\nu_e^2\right) - \mathbf{M}_a^1 \cdot \mathbf{g}_1\right)\mathbf{g}^1, \tag{6.14}$$

where $\mathbf{M}_a^1$ is the applied non-constraint moment. The mass matrix is invertible and a non-singular expression for the coordinate accelerations may be obtained. Utilizing the second equation of motion for $\ddot{\nu}^3$ and expression (6.7) for the dual Euler basis, one can show

$$\mathbf{M}_c^1 = \lambda^1\left(\dot{\nu}_e^2 \dot{f}\cos\left(\nu_e^2\right) - \dot{\nu}_e^2\dot{\nu}^3 + \ddot{f}\sin\left(\nu_e^2\right)\right)\mathbf{u} - \left(\mathbf{M}_a^1 \cdot \mathbf{u}\right)\mathbf{u}, \tag{6.15}$$

where

$$\mathbf{u} = \cos\left(\nu_e^2\right)\left(-\sin\left(f\right)\mathbf{E}_1 + \cos\left(f\right)\mathbf{E}_2\right) + \sin\left(\nu_e^2\right)\mathbf{E}_3. \tag{6.16}$$

The constraint moment, $\mathbf{M}_c^1 = \mu \mathbf{g}^1$ is well-defined for all ν_e^2, despite $\mathbf{g}^1$ becoming infinite at $\nu_e^2 = 0, \pi$. The basis $\{\mathbf{u}, \mathbf{g}_2, \mathbf{g}_3\}$ is orthonormal and thus suitable for capturing any applied moment that may contribute to the motion or the constraint moment. For example, to keep the cylinder in Figure 2(b) from rotating into the plane, a constraint moment in the direction $\mathbf{u} = \mathbf{e}_1 \times \mathbf{E}_3$ is required. If we chose to inherit $\nu^2 \in [0,\pi]$ from the parametrization of the ambient space, $\mathbb{R}P^3$, then two unilateral constraints are required: $\nu^2 \geq 0$ and $\nu^2 \leq \pi$. At these two values of ν^2, there is a constraint angular impulse of the form

$$\hat{\mathbf{M}}_c^1 = [[\mathbf{H}^1]] = \mathbf{H}^{1+} - \mathbf{H}^{1-} = \lambda^1\left(\boldsymbol{\omega}^{1+} - \boldsymbol{\omega}^{1-}\right), \tag{6.17}$$

where $\mathbf{H}^{1-}$ and $\mathbf{H}^{1+}$ are the angular momenta immediately before and after collision, respectively. The configuration manifold becomes $I \times S^1$, a manifold with two boundaries owing to the imposition of two unilateral constraints.

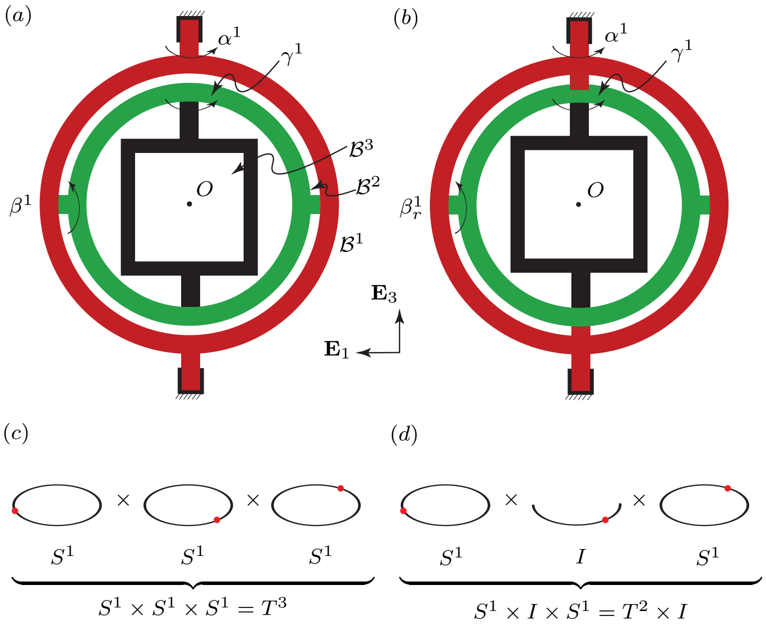

Fig. 8: *A system of (a) three nested gimbals has (c) the 3-torus, T^3, as its configuration manifold. When two unilateral constraints on the middle angle are imposed as in (b), the configuration manifold develops two boundaries and becomes (d) $T^2 \times I$.*

Consider a system of three rigid bodies, $\mathcal{B}^1, \mathcal{B}^2$, and $\mathcal{B}^3$ constrained in such a way as to have the reference configuration shown in Figure 8(a). It is sufficient for present purposes to assume all bodies to exhibit spherical symmetry so that the moment of inertia tensor about the center of mass of the Ith body is $\mathbf{J}^I = \lambda^I \mathbf{I}$. In the engineering design of inertial measurement units, $\mathcal{B}^3$ assumes the role of a stable platform while $\mathcal{B}^1$ and $\mathcal{B}^2$ are spherical gimbal mounts, and $\mathcal{B}^1$ is attached to a freely rotating spacecraft. This particular arrangement of gimbals is known as the Cardan suspension. In the subsequent derivation, we assume the space outside of the gimbals to be inertially fixed. Our analysis is similar to that found in [1, Section 11.8] for a rotating platform with vehicle fixed.

The position vector of the representative particle for the unconstrained system in $\mathcal{C}^{36}$ is

$$\mathbf{r} = \left(\bar{\mathbf{x}}^1, \bar{\mathbf{x}}^2, \bar{\mathbf{x}}^3, \mathbf{Q}^1, \mathbf{Q}^2, \mathbf{Q}^3\right). \tag{6.18}$$

Let $\left\{\alpha^1, \alpha^2, \alpha^3\right\}$ be a set of 3-2-1 Euler angles for $\mathbf{Q}^1$, $\left\{\beta^1, \beta^2, \beta^3\right\}$ a set of 1-3-2 Euler angles for the relative rotation tensor $\mathbf{Q}^2\left(\mathbf{Q}^1\right)^T$, and $\left\{\gamma^1, \gamma^2, \gamma^3\right\}$ a set of 3-2-1 Euler angles for the relative rotation tensor $\mathbf{Q}^3\left(\mathbf{Q}^2\right)^T$. If we use rectangular Cartesian coordinates for $\bar{\mathbf{x}}^I$, then the parametrizations of $\mathbf{Q}^I$ are independent of those for $\bar{\mathbf{x}}^I$, and we may focus our attention on the 27-dimensional subspace of $\mathcal{C}^{36}$ containing just the 3-tuples $\left(\mathbf{Q}^1,\mathbf{Q}^2,\mathbf{Q}^3\right)$. Denoting the Euler basis for each body as $\left\{\mathbf{g}_i{}^I\right\}$, the unconstrained angular velocity vectors are expressible as

$$\boldsymbol{\omega}^1=\sum_{i=1}^{3}\dot{\alpha}^i\mathbf{g}_i{}^1,\quad \boldsymbol{\omega}^2-\boldsymbol{\omega}^1=\sum_{i=1}^{3}\dot{\beta}^i\mathbf{g}_i{}^2,\quad \boldsymbol{\omega}^3-\boldsymbol{\omega}^2=\sum_{i=1}^{3}\dot{\gamma}^i\mathbf{g}_i{}^3. \tag{6.19}$$

The integrable constraints on the system are

$$\alpha^2=0,\quad \alpha^3=0,\quad \beta^2=0,\quad \beta^3=0,\quad \gamma^2=0,\quad \gamma^3=0. \tag{6.20}$$

A corotational basis for the Ith body is denoted $\left\{\mathbf{e}_i{}^I\right\}$, so that $\mathbf{e}_i{}^I=\mathbf{Q}^I\mathbf{E}_i$, while the dual Euler basis is denoted $\left\{\mathbf{g}^{iI}\right\}$. The constraints imply that the Euler basis and dual Euler basis for each body are orthonormal (though perhaps left-handed):

$$\begin{aligned}&\mathbf{g}_2{}^1=\mathbf{g}^{21}=\mathbf{e}_2{}^1,\quad \mathbf{g}_3{}^1=\mathbf{g}^{31}=\mathbf{e}_1{}^1,\quad \mathbf{g}_2{}^2=\mathbf{g}^{22}=\mathbf{e}_3{}^2,\\ &\mathbf{g}_3{}^2=\mathbf{g}^{32}=\mathbf{e}_2{}^2,\quad \mathbf{g}_2{}^3=\mathbf{g}^{23}=\mathbf{e}_2{}^3,\quad \mathbf{g}_3{}^3=\mathbf{g}^{33}=\mathbf{e}_1{}^3.\end{aligned} \tag{6.21}$$

The axes of suspension are

$$\mathbf{g}_1{}^1=\mathbf{g}^{11}=\mathbf{E}_3,\quad \mathbf{g}_1{}^2=\mathbf{g}^{12}=\mathbf{e}_1{}^2,\quad \mathbf{g}_1{}^3=\mathbf{g}^{13}=\mathbf{e}_3{}^3. \tag{6.22}$$

The constraints may be expressed in terms of the angular velocity vectors as

$$\begin{aligned}&\boldsymbol{\omega}^1\cdot\mathbf{e}_2{}^1=0,\quad \boldsymbol{\omega}^1\cdot\mathbf{e}_1{}^1=0,\quad \left(\boldsymbol{\omega}^2-\boldsymbol{\omega}^1\right)\cdot\mathbf{e}_3{}^2=0,\\ &\left(\boldsymbol{\omega}^2-\boldsymbol{\omega}^1\right)\cdot\mathbf{e}_2{}^2=0,\quad \left(\boldsymbol{\omega}^3-\boldsymbol{\omega}^2\right)\cdot\mathbf{e}_2{}^3=0,\quad \left(\boldsymbol{\omega}^3-\boldsymbol{\omega}^2\right)\cdot\mathbf{e}_1{}^3=0.\end{aligned} \tag{6.23}$$

Lagrange's prescription gives the constraint moments acting on each body about O as

$$\begin{aligned}&\mathbf{M}_c^1=\mu_1\mathbf{e}_2{}^1+\mu_2\mathbf{e}_1{}^1-\mu_3\mathbf{e}_3{}^2-\mu_4\mathbf{e}_2{}^2,\qquad \mathbf{M}_c^2=\mu_3\mathbf{e}_3{}^2+\mu_4\mathbf{e}_2{}^2-\mu_5\mathbf{e}_2{}^3-\mu_6\mathbf{e}_1{}^3,\\ &\mathbf{M}_c^3=\mu_5\mathbf{e}_2{}^3+\mu_6\mathbf{e}_1{}^3.\end{aligned} \tag{6.24}$$

The generalized coordinates are $\left\{q^1,q^2,q^3\right\}=\left\{\alpha^1,\beta^1,\gamma^1\right\}$, while the constrained coordinates are $\left\{q^4,\dots,q^9\right\}=\left\{\alpha^2,\alpha^3,\beta^2,\beta^3,\gamma^2,\gamma^3\right\}$. We note the following sets:

$$\begin{aligned}&\left\{\frac{\partial\boldsymbol{\omega}^1}{\partial\dot{q}^J}\right\}=\left\{\mathbf{E}_3,\mathbf{0},\mathbf{0},\mathbf{e}_2{}^1,\mathbf{e}_1{}^1,\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0}\right\},\qquad \left\{\frac{\partial\boldsymbol{\omega}^2}{\partial\dot{q}^J}\right\}=\left\{\mathbf{E}_3,\mathbf{e}_1{}^2,\mathbf{0},\mathbf{e}_2{}^1,\mathbf{e}_1{}^1,\mathbf{e}_3{}^2,\mathbf{e}_2{}^2,\mathbf{0},\mathbf{0}\right\},\\ &\left\{\frac{\partial\boldsymbol{\omega}^3}{\partial\dot{q}^J}\right\}=\left\{\mathbf{E}_3,\mathbf{e}_1{}^2,\mathbf{e}_3{}^3,\mathbf{e}_2{}^1,\mathbf{e}_1{}^1,\mathbf{e}_3{}^2,\mathbf{e}_2{}^2,\mathbf{e}_2{}^3,\mathbf{e}_1{}^3\right\}.\end{aligned} \tag{6.25}$$

The configuration manifold in this case is the 3-torus, T^3 as in Figure 8(c). A basis for the tangent space T_PT^3 is gotten as

$$\begin{aligned}&\mathbf{a}_1=\frac{\partial\mathbf{r}}{\partial\alpha^1}=\left(\operatorname{skwt}\left(\mathbf{E}_3\right)\mathbf{Q}^1,\frac{\partial\mathbf{Q}^2}{\partial\alpha^1},\frac{\partial\mathbf{Q}^3}{\partial\alpha^1}\right),\qquad \mathbf{a}_2=\frac{\partial\mathbf{r}}{\partial\beta^1}=\left(\mathbf{O},\frac{\partial\mathbf{Q}^2}{\partial\beta^1},\frac{\partial\mathbf{Q}^3}{\partial\beta^1}\right),\\ &\mathbf{a}_3=\frac{\partial\mathbf{r}}{\partial\gamma^1}=\left(\mathbf{O},\mathbf{O},\frac{\partial\mathbf{Q}^3}{\partial\gamma^1}\right).\end{aligned} \tag{6.26}$$

The set $\{\mathbf{a}_i\}$ is linearly independent for all values of $\mathsf{q} = \left[\alpha^1, \beta^1, \gamma^1\right]^T$, and the coordinate system is therefore singularity-free on T^3. The generalized forces are

$$
\begin{aligned}
Q_1 &= \left(\mathbf{M}_a^1 + \mathbf{M}_a^2 + \mathbf{M}_a^3\right) \cdot \mathbf{E}_3, \qquad Q_2 = \left(\mathbf{M}_a^2 + \mathbf{M}_a^3\right) \cdot {\mathbf{e}_1}^2, \qquad Q_3 = \mathbf{M}_a^3 \cdot {\mathbf{e}_3}^3, \\
Q_4 &= \left(\mathbf{M}_a^1 + \mathbf{M}_a^2 + \mathbf{M}_a^3\right) \cdot {\mathbf{e}_2}^1 + \mu_1, \qquad Q_5 = \left(\mathbf{M}_a^1 + \mathbf{M}_a^2 + \mathbf{M}_a^3\right) \cdot {\mathbf{e}_1}^1 + \mu_2, \\
Q_6 &= \left(\mathbf{M}_a^2 + \mathbf{M}_a^3\right) \cdot {\mathbf{e}_3}^2 + \mu_3, \qquad Q_7 = \left(\mathbf{M}_a^2 + \mathbf{M}_a^3\right) \cdot {\mathbf{e}_2}^2 + \mu_4, \\
Q_8 &= \mathbf{M}_a^3 \cdot {\mathbf{e}_2}^3 + \mu_5, \qquad Q_9 = \mathbf{M}_a^3 \cdot {\mathbf{e}_1}^3 + \mu_6.
\end{aligned} \tag{6.27}
$$

The constrained kinetic energy of the system is

$$
\begin{aligned}
T = T_2 = &\frac{1}{2}\lambda^1 \left(\dot{\alpha}^1\right)^2 + \frac{1}{2}\lambda^2 \left(\left(\dot{\alpha}^1\right)^2 + \left(\dot{\beta}^1\right)^2\right) \\
&+ \frac{1}{2}\lambda^3 \left(\left(\dot{\alpha}^1\right)^2 + \left(\dot{\beta}^1\right)^2 + \left(\dot{\gamma}^1\right)^2 + 2\dot{\alpha}^1\dot{\gamma}^1 \cos\left(\beta^1\right)\right),
\end{aligned} \tag{6.28}
$$

which is a non-degenerate positive-semidefinite quadratic form in $\dot{\mathsf{q}}$ for all q. The resulting equations of motion are

$$
\begin{bmatrix} \lambda^1 + \lambda^2 + \lambda^3 & 0 & \lambda^3 \cos\left(\beta^1\right) \\ 0 & \lambda^2 + \lambda^3 & 0 \\ \lambda^3 \cos\left(\beta^1\right) & 0 & \lambda^3 \end{bmatrix}
\begin{bmatrix} \ddot{\alpha}^1 \\ \ddot{\beta}^1 \\ \ddot{\gamma}^1 \end{bmatrix}
+ \lambda^3 \sin\left(\beta^1\right)
\begin{bmatrix} -\dot{\beta}^1\dot{\gamma}^1 \\ \dot{\alpha}^1\dot{\gamma}^1 \\ -\dot{\alpha}^1\dot{\beta}^1 \end{bmatrix}
= \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix}. \tag{6.29}
$$

For nonzero gimbal inertias λ^1 and/or λ^2, the mass matrix is invertible. When $\beta^1 = 0, \pi$, a moment applied to $\mathcal{B}^3$ in the ${\mathbf{e}_2}^2$-direction contributes nothing to Q_1, Q_2, or Q_3. Instead, it gets absorbed by $Q_4, \ldots, Q_9$ and contributes to the constraint moments. Gimbal lock occurs everywhere on the configuration manifold where $\beta^1 = 0$ and $\beta^1 = \pi$. As a consequence, the applied force $\mathbf{\Phi}_a$ on the representative particle corresponding to a nonzero applied moment $\mathbf{M}_a^3 = \left(\mathbf{M}_a^3 \cdot {\mathbf{e}_2}^2\right) {\mathbf{e}_2}^2$ is orthogonal to T^3.

When we allow $\lambda^1 = 0$ and $\lambda^2 = 0$, the configuration manifold degenerates from T^3 into $\mathbb{R}P^3$, the set $\left\{\alpha^1, \beta^1, \gamma^1\right\}$ becomes a set of ordinary extended 3-1-3 Euler angles for the rotation tensor of $\mathcal{B}^3$, and the singularities reappear. If one wishes to insert mechanical stops into the system as in Figure 8(b), then the equations of motion and constraint force prescriptions remain the same but a restricted value of $\beta_r^1 \in [0, \pi]$ is required. The unilateral constraints $\beta_r^1 \geq 0$ and $\beta_r^1 \leq 0$ would be enforced by constraint impulse moments, as previously discussed. The configuration manifold in that case develops two boundaries and becomes $T^2 \times I$, as shown in Figure 8(d).

6.4 Four Gimbals and an Elimination of Gimbal Lock

Consider now a system of four nested gimbals (Figure 9), where an additional gimbal $\mathcal{B}^0$ has been added to the mechanism analyzed in Section 6.3. The set of angles $\{\delta^1, \alpha^1, \beta^1, \gamma^1\}$ parametrize the constrained rotations of $\mathcal{B}^0$, $\mathcal{B}^1$, $\mathcal{B}^2$, and $\mathcal{B}^3$, respectively. Here, the configuration manifold is the 4-torus, T^4. If we further impose the constraint $\alpha^1 = \frac{\pi}{2}$, then it is clear that the three remaining suspension axes are constrained to be mutually orthogonal. The configuration manifold becomes T^3 and gimbal lock is avoided: there are no configurations where a nonzero moment applied to $\mathcal{B}^3$,

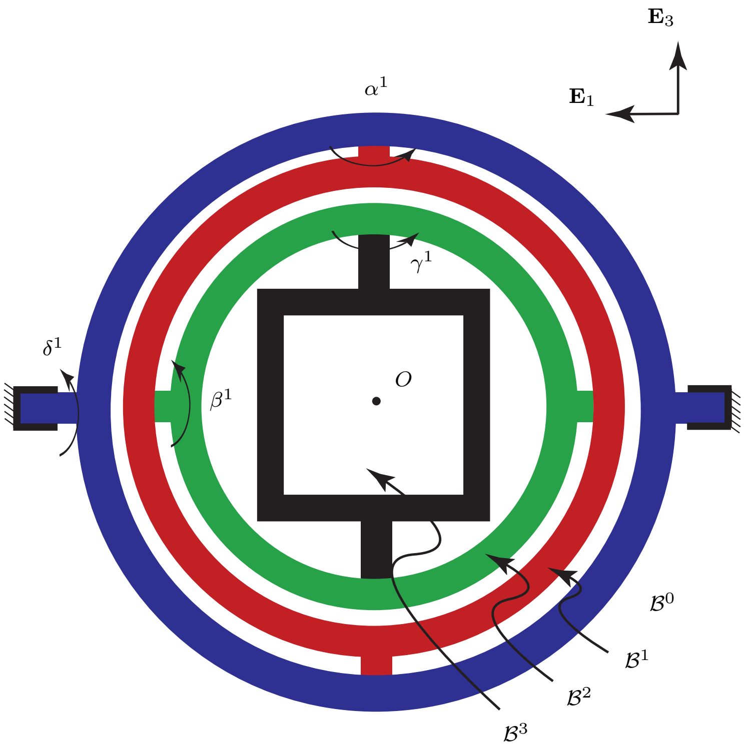

Fig. 9: *Four nested gimbals, under the constraint* $\alpha^1 = \frac{\pi}{2}$ *to ensure three mutually orthogonal suspension axes, has* T^3 *as its configuration manifold and is immune to gimbal lock.*

$\mathbf{M}_a^3$, corresponds to an orthogonal loading. In other words, a nonzero $\mathbf{M}_a^3$ always corresponds to a $\mathbf{\Phi}_a$ which has a component in the tangent space of T^3. This is clear if we compute Q_1, Q_2, and Q_3 for the unconstrained coordinates $\mathsf{q} = \left[\delta^1, \beta^1, \gamma^1\right]^T$:

$$Q_1 = \left(\mathbf{M}_a^1 + \mathbf{M}_a^2 + \mathbf{M}_a^3\right) \cdot \mathbf{E}_1, \qquad Q_2 = \left(\mathbf{M}_a^2 + \mathbf{M}_a^3\right) \cdot {\mathbf{e}_1}^2, \qquad Q_3 = \mathbf{M}_a^3 \cdot {\mathbf{e}_3}^3. \tag{6.30}$$

Owing to the constraint on α^1, $\{\mathbf{E}_1, {\mathbf{e}_1}^2, {\mathbf{e}_3}^3\}$ is a mutually orthogonal set and gimbal lock is nowhere to be found on the configuration manifold.

7 Conclusions

In this paper, we have offered a new perspective on coordinate singularities and gimbal lock using various examples from the dynamics of particles and rigid bodies. In particular, a coordinate singularity occurs when the covariant basis induced by the chosen coordinates fails to span the tangent space of the configuration manifold. We have demonstrated that a necessary condition for gimbal lock is that the system of applied forces and moments is orthogonal to the configuration manifold.

To remove a coordinate singularity, constraints can be imposed or additional inertias may be accounted for in the mechanical system. These actions serve to change the topology of the configuration manifold. In cases involving unilateral constraints, such as the particle sliding on a half-hoop as shown in Figure 5(b), the configuration manifold develops a boundary and constraint impulses are required to keep the representative particle on the manifold.

Using the classic example of a rigid body in a Cardan suspension, we have shown in Section 6.4 how gimbal lock is eliminated through the use of an additional fourth gimbal with an additional constraint. By making the three suspension axes mutually orthogonal, we ensure that there are no orthogonal loadings corresponding to a nonzero moment applied to the innermost body.

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A Appendix Background on Rotations

An example of a 3-dimensional configuration manifold results from the constrained motion of a rigid body rotating about a fixed point, such as in the motion of the Lagrange top. In this case, the rotation tensor $\mathbf{Q}$ of the top belongs to the proper-orthogonal subgroup of linear second-order tensors, a 3-dimensional manifold called Orth^+ embedded in $\mathcal{C}^9$. This subgroup is isomorphic to $SO(3)$, the special orthogonal group of 3×3 matrices. Furthermore, $SO(3)$ is diffeomorphic to $\mathbb{R}P^3$, real projective 3-space, and we can associate the configuration of the top to a point in $\mathbb{R}P^3$.[4]

Given an angle θ and axis of rotation $\mathbf{p}$, Euler's formula for the rotation tensor is

$$\mathbf{Q} = \mathbf{L}(\theta, \mathbf{p}) = \cos(\theta)(\mathbf{I} - \mathbf{p} \otimes \mathbf{p}) + \sin(\theta)\,\mathrm{skwt}(\mathbf{p}) + \mathbf{p} \otimes \mathbf{p}, \tag{A.1}$$

where $\mathrm{skwt}(\mathbf{p})$ indicates the skew tensor of $\mathbf{p}$, which is given by the mapping

$$\mathrm{skwt}(\mathbf{p}) = -\boldsymbol{\epsilon}\mathbf{p}, \tag{A.2}$$

where $\boldsymbol{\epsilon}$ is the third-order alternator tensor. Since $\mathbf{Q}$ is proper orthogonal, it satisfies $\mathbf{Q}\mathbf{Q}^T = \mathbf{I}$ and $\det \mathbf{Q} = +1$. A skew-symmetric angular velocity tensor may therefore be defined using $\mathbf{Q}$ as $\boldsymbol{\Omega} = \dot{\mathbf{Q}}\mathbf{Q}^T$. We denote the space of skew-symmetric tensors as Skw. The angular velocity vector $\boldsymbol{\omega}$ is the axial vector of $\boldsymbol{\Omega}$ which is given by the mapping

$$\boldsymbol{\omega} = \mathrm{ax}(\boldsymbol{\Omega}) = -\frac{1}{2}\boldsymbol{\epsilon}[\boldsymbol{\Omega}]. \tag{A.3}$$

[4] For representations of rotations with constant angular velocities on the real projective 2-space $\mathbb{R}P^2$, the reader is referred to [20].

Thus, there exists an isomorphism between Skw and $\mathbb{E}^3$ through the maps $\mathrm{ax}(\cdot)$ and $\mathrm{skwt}(\cdot)$ for which the following is true:

$$\mathbf{\Omega}\mathbf{a} = \boldsymbol{\omega} \times \mathbf{a} \tag{A.4}$$

for any $\mathbf{a} \in \mathbb{E}^3$. Euler angles are a common choice of parametrization for $\mathbf{Q}$. A general Euler angle decomposition is given by

$$\mathbf{Q} = \mathbf{L}\left(\nu^3, \mathbf{g}_3\right)\mathbf{L}\left(\nu^2, \mathbf{g}_2\right)\mathbf{L}\left(\nu^1, \mathbf{g}_1\right), \tag{A.5}$$

where $\left\{\nu^1, \nu^2, \nu^3\right\}$ are the Euler angles and $\{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\}$ are the corresponding axes of rotation, which are collectively known as the Euler basis [23,22,24]. It is also convenient to define a reciprocal basis $\left\{\mathbf{g}^i\right\}$ known as the dual Euler basis with the property

$$\mathbf{g}_i \cdot \mathbf{g}^j = \delta_i^j, \qquad (i, j = 1, 2, 3), \tag{A.6}$$

where δ_i^j is the Kronecker delta. A corotational (or body-fixed) basis is defined as the set $\{\mathbf{e}_i\}$ for which $\mathbf{e}_i = \mathbf{Q}\mathbf{E}_i$ for $i = 1, 2, 3$. The rotation tensor has the same components in the corotational basis as it does in the inertial basis: $\mathbf{Q} = Q_{ij}\mathbf{e}_i \otimes \mathbf{e}_j = Q_{ij}\mathbf{E}_i \otimes \mathbf{E}_j$. Using the abbreviations $\cos(\nu) = \mathrm{c}\nu$ and $\sin(\nu) = \mathrm{s}\nu$, a 3-2-1 Euler angle set yields the following parametrization:

$$[Q_{ij}] = \begin{bmatrix} \mathrm{c}\nu^1\mathrm{c}\nu^2 & \mathrm{c}\nu^1\mathrm{s}\nu^2\mathrm{s}\nu^3 - \mathrm{s}\nu^1\mathrm{c}\nu^3 & \mathrm{c}\nu^1\mathrm{s}\nu^2\mathrm{c}\nu^3 + \mathrm{s}\nu^1\mathrm{s}\nu^3 \\ \mathrm{s}\nu^1\mathrm{c}\nu^2 & \mathrm{s}\nu^1\mathrm{s}\nu^2\mathrm{s}\nu^3 + \mathrm{c}\nu^1\mathrm{c}\nu^3 & \mathrm{s}\nu^1\mathrm{s}\nu^2\mathrm{c}\nu^3 - \mathrm{c}\nu^1\mathrm{s}\nu^3 \\ -\mathrm{s}\nu^2 & \mathrm{c}\nu^2\mathrm{s}\nu^3 & \mathrm{c}\nu^2\mathrm{c}\nu^3 \end{bmatrix}, \tag{A.7}$$

where $\nu^1 \in [0, 2\pi)$, $\nu^2 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, and $\nu^3 \in [0, 2\pi)$. A 3-1-3 Euler angle set yields the following parametrization:

$$[Q_{ij}] = \begin{bmatrix} \mathrm{c}\nu^1\mathrm{c}\nu^3 - \mathrm{s}\nu^1\mathrm{c}\nu^2\mathrm{s}\nu^3 & -\mathrm{c}\nu^1\mathrm{s}\nu^3 - \mathrm{s}\nu^1\mathrm{c}\nu^2\mathrm{c}\nu^3 & \mathrm{s}\nu^1\mathrm{s}\nu^2 \\ \mathrm{s}\nu^1\mathrm{c}\nu^3 + \mathrm{c}\nu^1\mathrm{c}\nu^2\mathrm{s}\nu^3 & -\mathrm{s}\nu^1\mathrm{s}\nu^3 + \mathrm{c}\nu^1\mathrm{c}\nu^2\mathrm{c}\nu^3 & -\mathrm{c}\nu^1\mathrm{s}\nu^2 \\ \mathrm{s}\nu^2\mathrm{s}\nu^3 & \mathrm{s}\nu^2\mathrm{c}\nu^3 & \mathrm{c}\nu^2 \end{bmatrix}, \tag{A.8}$$

where $\nu^1 \in [0, 2\pi)$, $\nu^2 \in [0, \pi]$, and $\nu^3 \in [0, 2\pi)$.

For a given Euler angle parametrization, the following identity may be shown to hold:

$$\frac{\partial \mathbf{Q}}{\partial \nu^i}\mathbf{Q}^T = \mathrm{skwt}\left(\mathbf{g}_i\right), \quad (i = 1, 2, 3). \tag{A.9}$$

where $\mathbf{Q}$ is given by equation (A.5).

A.1 Constraining One of the Euler Angles

An interesting situation occurs when the first or third Euler angle is constrained and the configuration manifold for the rotational motion of the rigid body becomes a two-torus T^2 provided ν^2 is extended to $\nu_e^2 \in [0, 2\pi)$. The latter case arises in the motion of the rigid bodies shown in Figure 2.[5] If a set of 3-1-3 Euler angles are used to parameterize the rotation tensor $\mathbf{Q}$ of these bodies, then the third angle $\nu^3 = 0$. To use two angles to parameterize the rotation $\mathbf{Q}$ of the constrained rigid bodies shown in Figure 2, it is necessary that

$$\mathbf{Q} = \mathbf{L}\left(\nu_e^2, \mathbf{e}_1\right)\mathbf{L}\left(\nu^1, \mathbf{E}_3\right), \tag{A.10}$$

In contrast to the unconstrained case (A.5), it is necessary to extend the range of the second angle: $\nu^2 \to \nu_e^2 \in [0, 2\pi)$. The necessity of this extension can also be demonstrated by taking a pen and placing it on a horizontal surface. Fixing the orientation of the longitudinal axis of the pen is equivalent to fixing ν^1. Then, by rotating the pen about its longitudinal axis, the necessity of having the extended angle ν_e^2 can be readily seen.

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[5] A discussion of the constraint forces and moments acting on the bodies shown in this figure can be found in [24].

A.2 A Fixed Axis of Rotation

Suppose $\mathbf{Q}$ is constrained so that the rotation axis is fixed: $\mathbf{p} = \mathbf{p}_0$. Then, the configuration manifold is a 1-dimensional circle S^1 contained in Orth^+: $\mathbf{p}_0 \otimes \mathbf{p}_0$ is normal to the plane of the circle and $\cos(\theta)(\mathbf{I} - \mathbf{p}_0 \otimes \mathbf{p}_0) + \sin(\theta)\,\mathrm{skwt}(\mathbf{p}_0)$ is a radius vector to a point on the circle. A tensor spanning the tangent space to S^1 is

$$\mathbf{a}_1 = \frac{\partial \mathbf{Q}}{\partial \theta} = -\sin(\theta)(\mathbf{I} - \mathbf{p}_0 \otimes \mathbf{p}_0) + \cos(\theta)\ \mathrm{skwt}(\mathbf{p}_0)\,, \tag{A.11}$$

which is clearly tangent to the circle.